Math 21b Midterm 2 Review

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This material is meant to suggest topics to review. There are certainly other topics that may be covered on the exam, but these are a good place to start studying.

5 Orthogonality and Least Squares

5.1 Orthonormal Bases and Orthogonal Projections

- Orthogonal - two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$.
- Length or magnitude or norm - for a vector $\vec{v}$ in $\mathbb{R}^n$, this is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$. A vector with length 1 is called a *unit vector.*
- Orthonormal vectors - the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$ are called orthonormal if they are all unit vectors and orthogonal to one another. See page 178 for a neat way of saying this. Orthonormal vectors are linearly independent. If you have n orthonormal vectors in $\mathbb{R}^n$, you have a basis for $\mathbb{R}^n$.
- Orthogonal complement - V is a subspace of $\mathbb{R}^n$ with basis $\vec{v}_1, \ldots, \vec{v}_m$. $V^\perp = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{v}_i \cdot \vec{x} = 0, \text{ for } i = 1, \ldots, m\}$. $V^\perp$ is also a subspace of $\mathbb{R}^n$.
- Orthogonal projection - V a subspace of $\mathbb{R}^n$ with orthonormal basis $\vec{v}_1, \ldots, \vec{v}_m$. For any $\vec{x}$ in $\mathbb{R}^n$, there is a unique vector $\vec{w}$ in V such that $\vec{x} - \vec{w}$ is in $V^\perp$. This $\vec{w}$ is the *orthogonal projection* of $\vec{x}$ onto V, denoted $\text{proj}_V \vec{x}$, and

  $$\text{proj}_V \vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \ldots + (\vec{v}_m \cdot \vec{x})\vec{v}_m.$$

  Also, if $\vec{v}_1, \ldots, \vec{v}_n$ is an orthonormal basis for $\mathbb{R}^n$, then for all $\vec{x}$ in $\mathbb{R}^n$,

  $$\vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \ldots + (\vec{v}_m \cdot \vec{x})\vec{v}_m.$$

- Angle - the angle α between two nonzero vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ is

  $$\alpha = \arccos \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|}.$$

5.2 Gram-Schmidt Process and QR Factorization

- Gram-Schmidt Process - a way of making a set of linearly independent vectors into an orthonormal set of vectors without changing the actual space they span. The steps for this process can be found on page 197 of the text, and essentially amount to repetition of these two steps for each vector:

  1. Make the vector perpendicular to all the other vectors already looked at.
  2. Make it a unit vector.

- QR Factorization - a way of storing the information used (lengths and projections) in the Gram-Schmidt Process in two matrices. The exact content of each matrix can be found on page 198 of the text.

5.3 Orthogonal Transformations and Orthogonal Matrices

- Orthogonal transformations - a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ preserving the length of vectors: $\|T(\vec{x})\| = \|\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$. Orthogonal transformations preserve orthogonality.

- Orthogonal matrices - the matrix A where $T(\vec{x}) = A\vec{x}$ is an orthogonal transformation.

- Properties:

  - A linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ is orthogonal if and only if the vectors $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ form an orthonormal basis for $\mathbb{R}^n$.
  - An n x n matrix A is orthogonal if and only if its columns form an orthonormal basis of $\mathbb{R}^n$.
  - The product of two orthogonal n x n matrices is orthogonal.
  - The inverse of an orthogonal n x n matrix is orthogonal.

- Transpose - the transpose A^T of a matrix A reflects the elements of A across the main diagonal. In other words, elements a_{ij} and a_{ji} are interchanged. Notice that we can now write $v \cdot w$ as $v^T w$.

- Symmetric - a matrix A is called symmetric if $A^T = A$. (Notice that a symmetric matrix must be a square matrix.) The transpose has the following properties:

  1. $(AB)^T = B^T A^T$ (note that the order switches!).
  2. $(A^{-1})^T = (A^T)^{-1}$.
  3. rank $A =$ rank A^T.

- Skew-symmetric - a matrix A is called skew-symmetric if $A^T = -A$. (Notice that a skew-symmetric matrix must be a square matrix.)

- An n x n matrix A is orthogonal if and only if $A^T A = I_n$. This is the same as saying that $A^{-1} = A^T$.

- The following statements are equivalent (all true or all false) for an n x n matrix A:

  1. A is an orthogonal matrix.
  2. The transformation $L(\vec{x}) = A\vec{x}$ preserves length, that is, $\|A\vec{x}\| = \|\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$.
  3. The columns of A form an orthonormal basis of $\mathbb{R}^n$.
  4. $A^T A = I_n$.
  5. $A^{-1} = A^T$.

- Orthogonal projection onto V - if V is a subspace of $\mathbb{R}^n$ with orthonormal basis $\vec{v}_1, \ldots, \vec{v}_m$ and A is the matrix whose columns are $\vec{v}_1, \ldots, \vec{v}_m$, then the matrix representing the orthogonal projection onto V is AA^T (note the order).

5.4 Least Squares and Data Fitting

- Characterization of orthogonal complements - $(imA)^{\perp} = ker(A^T)$.

- For a subspace V of $\mathbb{R}^n$,

  1. $\dim(V) + \dim(V^{\perp}) = n$.
  2. $(V^{\perp})^{\perp} = V$.
  3. $V \cap V^{\perp} = \{\vec{0}\}$.

- If A is an m x n matrix, then $ker(A) = ker(A^T A)$. If $ker(A) = \{\vec{0}\}$, then $A^T A$ is invertible.

- Read the discussion of least squares fitting and the normal equation (page 213-221).

6 Determinants

6.1 Introduction to Determinants

- Determinant - only square matrices have determinants, it is not defined for a nonsquare matrix. The determinant of a matrix can be thought of in two ways: as a map from the set of square matrices to the real numbers, or as a real number associated with a matrix. The more correct way to view it is probably as a map, but either way is acceptable.

- Determinant of 1 x 1 - if the matrix is $\begin{bmatrix} m \end{bmatrix}$, then the determinant is just m.

- Determinant of 2 x 2 - if the matrix is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the determinant is $ad - bc$.

- Determinant of 3 x 3 - if the columns of the matrix are $\vec{u}$, $\vec{v}$, and $\vec{w}$, then the determinant is $\vec{u} \cdot (\vec{v} \text{ x } \vec{w})$. Another way to find the determinant is described on page 243 of the text.

- The determinant of a triangular matrix is the product of the diagonal entries of the matrix.

6.2 Properties of the Determinant

- Properties of the determinant (A is a square matrix):

  1. $\det(A^T) = \det(A)$.
  2. The determinant is linear in the columns.
  3. $\det(AB) = \det(A)\det(B)$.
  4. $\det(A^{-1}) = (\det(A))^{-1}$.

- Effects of row operations on the determinant, where A is an n x n matrix and B is the matrix obtained by performing some operation on A: If the operation is swapping two rows, $\det(B) = -\det(A)$. If the operation is dividing a row of A by k, then $\det(B) = (1/k)\det(A)$. If the operation is adding a scalar multiple of one row to another, then $\det(B) = \det(A)$. This can be summarized by saying that in the process of row reducing A to the identity matrix, if $s = \#$ of swaps made, and if $k_1, \ldots, k_r$ are the scalars that individual rows were divide by, then $\det(A) = (-1)^s k_1 k_2 \cdots k_r$.

- An n x n matrix A is invertible if and only if $\det(A) \neq 0$.

- One way to compute the determinant is by Laplace expansion (see page 262).

6.3 Geometrical Interpretations of the Determinant; Cramer's Rule

- Rotation matrix - an orthogonal n x n matrix with determinant 1 is called a rotation matrix.

- Parallelepiped - if $\vec{v}_1, \ldots, \vec{v}_n$ are vectors in $\mathbb{R}^n$, then the area of the parallelepiped defined by them is the determinant of the matrix whose column vectors are $\vec{v}_1, \ldots, \vec{v}_n$. If you only have vectors $\vec{v}_1, \ldots, \vec{v}_k$ in $\mathbb{R}^n$ where $k \leq n$, the volume of the k-parallelepiped is $\sqrt{\det(A^T A)}$ where A is the matrix whose columns are $\vec{v}_1, \ldots, \vec{v}_k$. In the case where $k = n$, this is exactly the same as the first definition.

- Cramer's Rule - to solve the system $A\vec{x} = \vec{b}$, the i^{th} coordinate of the vector $\vec{x}$, is $\det(A_i)/\det(A)$ where A_i is the matrix formed by replacing the i^{th} column of A by the vector $\vec{b}$.

7 Eigenvalues and Eigenvectors

7.1 Dynamical Systems and Eigenvectors: An Introductory Example

- State vector - a vector that contains all the information of what is happening in a system at a time t. (See the roadrunner example.)

- Phase portrait - a picture of what happens at present, past, and future times. It plots the vectors $x(t)$ for all integers t.

- Eigenvector - an eigenvector of the matrix A is a nonzero vector $\vec{v}$ such that $A(\vec{v}) = \lambda\vec{v}$ for some value λ.

- Eigenvalue - the value λ referred to above.

- The only possible eigenvalues of an orthogonal matrix are ± 1.
- Very important to study - go through Summary 7.1.4 on page 299. Memorize it if necessary, but make sure you understand roughly why each of the statements are equivalent.

7.2 Finding the Eigenvalues of a Matrix

- For an n x n matrix A and a scalar λ, λ is an eigenvalue of A if and only if $\det(\lambda I_n - A) = 0$.
- The eigenvalues of a triangular matrix are its diagonal entries.
- Characteristics polynomial - $f_A(\lambda) = \det(\lambda I_n - A)$. The eigenvalues of an n x n matrix A are the zeroes of this function. The coefficient of λ^{n-1} in the characteristic polynomial is $-tr(A)$, and the constant term is $(-1)^n \det(A)$.
- Algebraic multiplicity - an eigenvalue λ_0 has algebraic multiplicity k if $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$ for a polynomial $g(\lambda)$ that λ_0 is not a root of. In other words, λ_0 is a root of multiplicity k of $f_A(\lambda)$.
- An n x n matrix has at most n eigenvalues, even if they are counted with their algebraic multiplicities. If n is odd, then an n x n matrix has at least one eigenvalue.

7.3 Finding the Eigenvectors of a Matrix

- Eigenspace - the kernel of the matrix $\lambda I_n - A$ is the eigenspace associated with λ, and written E_λ. This space consists of all solutions $\vec{v}$ to the equation $A(\vec{v}) = \lambda \vec{v}$.
- Geometric multiplicity - if λ is an eigenvalue of a matrix A, then the dimension of E_λ is the geometric multiplicity of λ. (Geometric multiplicity of λ) $= \dim(E_\lambda) \leq$ (algebraic multiplicity of λ).
- Eigenbasis - if A s an n x n matrix, a basis of $\mathbb{R}^n$ consisting of eigenvectors of A is an eigenbasis for A.
- If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are eigenvectors of an n x n matrix A with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, then the vectors are linearly independent. If A has n distinct eigenvalues then there is an eigenbasis for A found by choosing an eigenvector for each eigenvalue. If the geometric multiplicities of the eigenvalues of A add up to n then there is an eigenbasis for A found by choosing a basis of each eigenspace and combining these vectors.
- Eigenvalues of similar matrices A and B:
  1. $f_A(\lambda) = f_B(\lambda)$.
  2. $\text{rank}(A) = \text{rank}(B)$ and $\text{nullity}(A) = \text{nullity}(B)$.
  3. A and B have the same eigenvalues, with the same algebraic and geometric multiplicities. (Note: the eigenvectors need not be the same.)
  4. $\det(A) = \det(B)$ and $tr(A) = tr(B)$.

7.4 Diagonalization

- The matrix of a linear transformation with respect to an eigenbasis is the diagonal matrix with each diagonal entry as the eigenvalue for the corresponding eigenvector.

- Diagonalizable - an n x n matrix A is diagonalizable if A is similar to a diagonal matrix D. That is, if there is an invertible n x n matrix S with $D = S^{-1}AS$ diagonal. A matrix is diagonalizable if and only if there is an eigenbasis for it. So if an n x n matrix has n distinct eigenvalues, it is diagonalizable.

- To determine if an n x n matrix is diagonalizable,

  1. Find its eigenvalues (solve the characteristic equation).
  2. Find a basis of the eigenspace corresponding to each eigenvalue.
  3. The matrix is diagonalizable if and only if the dimensions of the eigenspaces add up to n. Then an eigenbasis can be found by combining the bases of the eigenspaces found previously.

- Powers of a diagonalizable matrix - if a matrix A is diagonalizable, we can find an invertible S such that $D = S^{-1}AS$ is diagonal. So $A = SDS^{-1}$. Then $A^t = SD^tS^{-1}$, and D^t looks like D except with each of the diagonal entries raised to the t^{th} power.

- We can extend the ideas of eigenvalues and eigenvectors to linear transformations on linear spaces (not just on vector spaces).

7.5 Complex Eigenvalues

- Complex numbers - $z = a + bi$ where $i = \sqrt{-1}$. The "real part" of z is a, and b is the "imaginary part" of z. The length of z, $|z|$, is called the modulus of z, and the angle z makes, ϕ, with the positive horizontal axis is called the argument of z. Complex numbers operate in the following ways:

$$\begin{gathered}(a+bi)+(c+di)=(a+c)+(b+d)i.\\ (a+bi)(c+di)=(ac-bd)+(ad+bc)i.\\ |zw|=|z|\ |w|.\\ \arg(zw)=\arg z+\arg w \quad (\text{modulo } 2\pi).\end{gathered}$$

- Polar form - the polar form of the complex number z is $z = r(\cos\phi + i\sin\phi)$.

- De Moivre's formula - $(\cos\phi + i\sin\phi)^n = \cos(n\phi) + i\sin(n\phi)$.

- Fundamental theorem of algebra - any polynomial $p(\lambda)$ with complex coefficients can be written as a product of linear factors:

$$p(\lambda) = k(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

  for some complex numbers (not necessarily distinct) $\lambda_1, \ldots, \lambda_n, k$. So a polynomial of degree n has exactly n roots when counted with multiplicity.

- Complex eigenvalues and eigenvectors - a complex n x n matrix has n complex eigenvalues if eigenvalues are counted with their algebraic multiplicities. If the eigenvalues are $\lambda_1, \ldots, \lambda_n$, listed with their algebraic multiplicities, then $tr(A) = \lambda_1 + \ldots + \lambda_n$ and $\det(A) = \lambda_1 \cdots \lambda_n$.

7.6 Stability

- Stable equilibrium - in a dynamical system $\vec{x}(t+1) = A\vec{x}(t)$, $\vec{0}$ is a stable equilibrium if $\lim_{t\to\infty} \vec{x}(t) = \vec{0}$. This is true if and only if the modulus of all complex eigenvalues of A is less than 1. If the modulus is 1, the trajectory is an ellipse, and if it is greater than one then the trajectory spirals outwards.
- Review Fact 7.6.3 on page 358.

9 Linear Differential Equations

9.1 An Introduction to Continuous Dynamical Systems

- Exponential growth and decay - a linear differential equation $\frac{dx}{dt} = kx$ with initial value x_0 has solution $x(t) = e^{kt}x_0$.
- If $\vec{x}(t+1)$ depends linearly on $\vec{x}(t)$, we can write $\vec{x}(t+1) = A\vec{x}(t)$, or $\vec{x}(t) = A^t\vec{x}_0$.
- Understand the difference between a discrete system and a continuous one. (See solution to continuous dynamical systems - Fact 9.1.2 on page 406.)

9.2 The Complex Case: Euler's Formula

- Complex exponential functions - if λ is a complex number, then $z = e^{\lambda t}$ is the unique complex-valued function such that $\frac{dz}{dt} = \lambda z$ and $z(0) = 1$.
- Euler's formula - $e^{it} = \cos(t) + i\sin(t)$. So from the polar representation of z, we can write $z = re^{i\phi}$.
- A continuous dynamical system has stability in the zero state if and only if the real parts of all eigenvalues of the corresponding matrix are negative. If the matrix is a 2 x 2 matrix, the zero state is asymptotically stable if and only if $tr(A) < 0$ and $\det(A) > 0$.
- Review Fact 9.2.6 on page 417.
- Review Figure 10 on page 420.