Practice Problems: Test #2

Please note: *I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on any test will resemble these problems in any way whatsoever.*

Remember: *On exams, you have to supply evidence for your conclusions, and explain why your answers are appropriate.*

1. A square matrix A is called *symmetric* if $A = A^T$, and skew-symmetric if $A = -A^T$.

(a) Suppose that S is a symmetric matrix and K is a skew-symmetric matrix. Under what conditions is it possible for S to equal K?

(b) Show that the only diagonal skew-symmetric matrix is the matrix with zeros in every entry.

(c) Suppose that if A and B are both symmetric matrices of the same size. Show that the sum of A and B is also a symmetric matrix.

(d) Suppose that if A and B are both symmetric matrices of the same size. Is it possible that AB is also a symmetric matrix? If so, what condition is necessary for AB to be a symmetric matrix?

2. If A is a 3 by 3 matrix whose determinant is 5, and if $B = QAP$, where:

$$Q = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 0 & 4 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix},$$

what is the determinant of B?

3. Let V denote the set of solutions of the homogeneous differential equation:

$$y''' - 3y'' + y' - 3y = 0.$$

It is possible to show (although you are not required to show here) that V is a vector space, and that $S = \{e^{3x}, \cos(x), \sin(x)\}$ is a basis for V.

(a) Verify that $f_1(x) = 2e^{3x}$, $f_2(x) = \cos(x) + \sin(x)$, and $f_3(x) = e^{3x} - \sin(x)$ are contained in V, and find the coordinates with respect to the basis S for each of these functions.

(b) Show that $W = \{f_1, f_2, f_3\}$ is a basis for V.

4. Find the characteristic polynomials and eigenvalues of the following matrices:

(a) $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 3 & -7 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 \\ -16 & 10 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 5 & 1 & -4 \end{bmatrix}$

5. In this problem, the matrix will be:

$$A = \begin{bmatrix} -1 & -1 & 1 & -1 \\ -2 & 0 & 1 & -1 \\ -2 & 3 & -2 & -1 \\ 4 & -10 & 10 & 3 \end{bmatrix}.$$

(a) Sketch a graph of the characteristic polynomial of A, and use this graph to decide how many distinct eigenvalues A has.

(b) What are the distinct eigenvalues of A?

(c) Find bases for the eigenspaces belonging to each of the eigenvectors you found in part (b).

(d) Do the eigenvectors of A form a basis for $\boldsymbol{R}^4$ or not? Explain your reasoning.

6. In each of the following cases, decide whether the collection of functions is a subspace of the vector space of real-valued functions of a single variable.

(a) All functions $f(x)$ such that: $\left| \int_0^1 f(x)dx \right| < \infty$.

(b) All solutions of the differential equation: $y'' - (\sin(x))y' + e^x y = 0$.

(c) All functions $f(x)$ with the property that: $f(x) \geq f(y)$ if $x \geq y$.

(d) All functions $f(t)$ that satisfy: $\int_0^1 f(t) \cdot \sin(t)dt = 0$.

(e) All functions that satisfy $f(x) \geq 0$ for all real numbers x.

7. Let A be an n by n matrix, and let γ be a non-zero vector.

(a) Show that $\{\gamma, A\gamma, \ldots, A^n\gamma\}$ is a linearly dependent set.

(b) Show that the subspace Span$\{\gamma, A\gamma, \ldots, A^n\gamma\}$ has dimension 1 if and only if γ is an eigenvector of A.

8. Let A be a 4 by 4 matrix with the following properties:

(I) Two of the eigenvalues of A are: 3 and 2.

(II) 3 is an eigenvalue of the matrix $A + 2I_4$.

(III) $\det(A) = 12$.

(a) What are the other two eigenvalues of A ?

(b) What is the characteristic polynomial of A ?

(c) What is the charateristic polynomial of A^{-1} ?

9. Suppose that the set of vectors $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ is an orthonormal basis for $\boldsymbol{R}^n$. Let $a_1, \ldots, a_n$ be any real scalars, and define:

$$A = a_1\boldsymbol{u}_1\boldsymbol{u}^T_1 + \ldots + a_n\boldsymbol{u}_n\boldsymbol{u}^T_n.$$

(a) What sort of algebraic object (scalar, vector, matrix) is A ?

(b) What is the relationship between A and its transpose ?

(c) Show that $a_1, \ldots, a_n$ are the eigenvalues of A.

(d) What are the eigenvectors of A ?

10. Suppose that A is an n by n matrix that is diagonalizable. Show that each of the following matrices is also diagonalizable.

(a) sA, where s is a scalar.

(b) A^T.

(c) A^{-1}, provided that A is non-singular.

11. In this problem, the matrix A will always refer to:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

(a) Find the characteristic equation of A, $p_A(x)$.

(b) It is possible to substitute a matrix into a polynomial equation in much the same way as you can substitute a numerical value into a polynomial equation. The main difference is that the constant term of the polynomial must be replaced by a scalar multiple of the identity matrix. For example, consider the polynomial function:

$$f(x) = 4x^2 - 9x + 11.$$

Then $f(A)$ would be the matrix:

$$f(A) = 4 \cdot \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} - 9 \cdot \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + 11 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If you work all of this out, then you just get a 2 by 2 matrix as the result. Use the characteristic polynomial from Part (a) to calculate the 2 by 2 matrix: $p_A(A)$.

(c) Let B be an n by n diagonalizable matrix. Show that A satisfies its own characteristic equation.

12. Find the distance from the point (1, -1, 1, 1) to the hyperplane (i.e. higher dimensional equivalent of a plane) spanned by:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

13. (a) Suppose that $A = PDP^{-1}$, where P is a 2 by 2 matrix and $D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$. Define a 2 by 2 matrix: $B = 5I - 3A + A^2$. Is B diagonalizable?

(b) Let $\boldsymbol{u}$ be a unit vector from $\boldsymbol{R^n}$. Define a matrix $Q = I_n - 2\boldsymbol{uu}^T$. Show that Q is an orthogonal matrix.

14. Let A be a diagonalizable 3 by 3 matrix whose characteristic polynomial is:

$$p_A(x) = x^3 - 3x^2 + 4.$$

(a) If $q(x)$ is a polynomial with the properties that $q(-1) = q(2) = 0$, show that $q(A) = 0$.

(b) Find the polynomial, m, of smallest degree for which $m(A) = 0$.

(c) Express each of the matrices: A^3, A^4, A^5 and A^{-1} as a linear combination of A and the 3 by 3 identity matrix, I.

15. In this problem, the matrix A will always be:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) If A was the coefficient matrix in a system of differential equations, would the origin, (0, 0), be an asymptotically stable equilibrium?

(c) Suppose that A is the coefficient matrix for a 2 by 2 system of differential equations:

$$\frac{d\vec{x}}{dt} = A \cdot \vec{x}$$

with initial value $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find an explicit formula for the solution vector $\vec{x}(t)$.

Answers

1.(a) The two matrices must be the same size and the two matrices must each have zeros in every entry.

1.(b) Suppose that H is a diagonal, skew-symmetric matrix. Because H is diagonal, every entry except the ones of the form h_{ij} is zero. Because $H = -H^T$, every diagonal entry must satisfy $h_{jj} = -h_{jj}$. So all of the diagonal entries are zero as well.

1.(c) $(A + B)^T = A^T + B^T = A + B$.

1.(d) The matrices have to commute.

2 . $\det(B) = -425$.

3.(a) To show that f_1, f_2 and f_3 are contained in V, it is sufficient to show that they can be written as linear combinations of vectors from the basis S. It is pretty clear that this is the case, and the coordinates of f_1 are (2, 0, 0), the coordinates of f_2 are (0, 1, 1) and the coordinates of f_3 are (1, 0, −1).

3.(b) One way to do this is to show that the linear mapping $T:V \rightarrow V$ defined by:

$$T(e^x) = f_1(x) \qquad T(\cos(x)) = f_2(x) \qquad T(\sin(x)) = f_3(x)$$

is an isomorphism. Writing everything in terms of the coordinates in the basis S, the linear transformation T can be represented by the 3 by 3 matrix:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

This is an invertible matrix, so $T:V \rightarrow V$ is an isomorphism. Since the isomorphic image of a basis is also a basis, then the given set of vectors is a basis for V.

4.(a) $x^3 - 14x^2 + 64x - 96$. Eigenvalues are 4 (multiplicity 2) and 6.

4.(b) $x^2 - 12x + 36$. Eigenvalue is 6 (multiplicity 2).
4.(c) $x^3 + 2x^2 - 8x$. Eigenvalues are 0, -4 and 2.

5.(a) The graph of the characteristic polynomial is a quartic with two roots. There are exactly two eigenvalues of A.
5.(b) The eigenvalues are 1 and -3.
5.(c) The eigenspace belonging to 1: {(-1, 0, 0, 2), (0, 1, 1, 0)}.
The eigenspace belonging to -3: {(1, 1, 0, 1)}.
5.(d) The eigenvectors are linearly independent, but do not span all of $\boldsymbol{R}^4$.

6.(a) yes
6.(b) yes
6.(c) no
6.(d) yes
6.(e) no

7.(a) The most immediate way to see this is to note that since A is an n by n matrix, this must be a set of n-dimensional vectors. There are $n + 1$ vectors in the set. No set of more than n n-dimensional vectors can be linearly independent.
7.(b) Suppose that the subspace spanned by those vectors has dimension 1. Then all of the vectors in the set must be scalar multiples of j. In particular Aj is a scalar multiple of j, so j must be an eigenvector of A. Next, suppose that j is an eigenvector of A. Then A^pj is a scalar multiple of j for every positive integer power p. Therefore, every vector in the set is a scalar multiple of j, and the subspace of $\mathbf{R}^n$ spanned by the set of vectors is equal to Span{j}. Therefore, the dimension of the subspace spanned by the set of vectors is equal to 1.

8.(a) The other two eigenvalues are 1 and 2.
8.(b) Characteristic polynomial = $(x - 1)(x - 2)^2(x - 3)$.
8.(c) The eigenvalues of A^{-1} are 1, 1/2, 1/2 and 1/3. The characteristic polynomial of A^{-1} is:
$(x - 1)(x - 1/2)^2(x - 1/3)$.

9.(a) A is an n by n matrix.
9.(b) A is equal to its transpose.
9.(c) Multiply each side on the right by $\boldsymbol{u}_j$, and then use orthonormality to get: $A\boldsymbol{u}_j = a_j\boldsymbol{u}_j$.
9.(d) From the working in part (c), the eigenvectors are $\boldsymbol{u}_j$, $j = 1, ..., n$.

10. We have: $A = PDP^{-1}$. So:
10.(a) $sA = P(sD)P^{-1}$.
10.(b) $A^T = (P^T)^{-1}DP^T$.
10.(c) $A^{-1} = P^{-1}D^{-1}P$.

11.(a) The characteristic equation is: $(2 - x)^2 - 3 = 0$.
11.(b) The result for $p_A(A)$ is the 2 by 2 zero matrix.
11.(c) $A = PDP^{-1}$, where D is the diagonal matrix with the eigenvalues of A along the main diagonal. Let $p_A(x)$ denote the characteristic polynomial of A. Then it is possible to show that:

$$p_A(A) = P\begin{bmatrix} p_A(\lambda_1) & 0 & \cdots & 0 \\ 0 & p_A(\lambda_2) & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & p_A(\lambda_n) \end{bmatrix}P^{-1}$$

where λ_j are the eigenvalues of A. Since the eigenvalues are the solutions of $p_A(x) = 0$, the matrix $p_A(A)$ will be the n by n zero matrix.

12. The distance is $(2/3)^{1/2}$. (Do the orthogonal projection of the vector onto the subspace.)

13.(a) Yes.
13.(b) Need to show that: $Q^TQ = I_n$. Now, $Q^T = I - 2uu^T = Q$. So:

$Q^TQ = I - 4uu^T + 4uu^Tuu^T$.

Since $u^Tu = 1$ (u is a unit vector), the last two terms cancel.

14.(a) Know from problem 11 that a matrix satisfies its own characteristic polynomial. Can factor the given characteristic polynomial as: $p_A(x) = (x + 1)(x - 2)^2$. This means that:

$(A + I_n)(A - 2I_n)(A - 2I_n)$ is equal to the zero matrix.

If we were dealing with ordinary multiplication, then the equation given above would imply that (at least) one of the linear factors was equal to the zero matrix. However, in matrix multiplication it is possible for two non-zero matrices to multiply together to give the zero matrix as the result so the argument is not quite so straight-forward here. Since A is diagonalizable, there is an invertible matrix S that satisfies the relationship:

$$SAS^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Observe that:

$$S(A + I_n)(A - 2I_n)S^{-1} = S(A + I_n)S^{-1}S(A - 2I_n)S^{-1} = (SAS^{-1} + I_n)(SAS^{-1} - 2I_n).$$

So that:

$$S(A + I_n)(A - 2I_n)S^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Multiplying on both sides of this equation by S and S^{-1} gives that $(A + I_n)(A - 2I_n)$ is equal to the zero matrix. Given the information about $q(x)$, we can infer that: $q(x) = (x + 1)(x - 2)f(x)$, where $f(x)$ is a polynomial. Since $p_A(A)$ = the zero matrix, then since the matrix A is diagonalizable, $(A + I)(A - 2I)$ is the zero matrix and $q(A)$ will be the zero matrix as well.
14.(b) $m(x)$ will consist of the distinct factors of $p_A(x)$, i.e. $m(x) = (x + 1)(x - 2)$.

14.(c) Since $m(A) = 0$, have that: $A^2 = A + 2I$. Use this in the other powers to get:
$A^3 = 3A + 2I$. $A^4 = 5A + 6I$. $A^5 = 11A + 10I$. $A^{-1-} = 0.5A - 0.5I$.

15.(a) Eigenvalues are $4\pm\sqrt{15}$. Possible eigenvectors (belonging to $4+\sqrt{15}$ and $4-\sqrt{15}$ respectively): $v_1 = (2, 3+\sqrt{15})$ and $v_2 = (2, 3-\sqrt{15}.)$.

15.(b) As the moduli of the eigenvalues are greater than one, the point (0, 0) will not be an asymptotically stable equilibrium.

15.(c) The explicit formula for the solution vector is:

$$\vec{x}(t) = 0.0563 \cdot e^{\left(4+\sqrt{15}\right)t} \cdot \begin{bmatrix} 2 \\ 3+\sqrt{15} \end{bmatrix} + 0.4436 \cdot e^{\left(4-\sqrt{15}\right)t} \cdot \begin{bmatrix} 2 \\ 3-\sqrt{15} \end{bmatrix}.$$