Practice Problems: Final Exam

Please note: *I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on any test will resemble these problems in any way whatsoever.*

Remember: *On exams, you have to supply evidence for your conclusions, and explain why your answers are appropriate.*

1. A continuous, 2π-periodic function $f(x)$ has the Fourier series expansion,

$$\sum_{n=1}^{\infty} \tfrac{1}{3n^2} \cdot \sin(n \cdot x).$$

Evaluate each of the following integrals and carefully justify your evaluation.

(a) $\int_{-\pi}^{\pi} f(x)dx$

(b) $\int_{0}^{\pi} f(x)\cdot\sin(3x)dx$

(c) $\int_{-\pi}^{\pi} f(x)\cdot\cos(5x)dx$

(d) $\int_{-\pi}^{\pi} f(x)\cdot\cosh(2x)dx$

2. Find a solution $u(x, t)$ of the initial/boundary value problem,

- $4\cdot\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$
- $u(0, t) = 0$ and $u(\pi, t) = 0$ for $t > 0$.
- $u(x, 0) = 0$ for $0 \le x \le \pi$.
- $\dfrac{\partial u}{\partial t}(x,0) = 2\cdot\sin(3x)$ for $0 \le x \le \pi$.

3. Find the solution $u(x, t)$ of the initial/boundary value problem,

- $9\cdot\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ for $0 < x < 4$ and $t > 0$.
- $u(0, t) = 0$ and $u(4, t) = 0$ for $t > 0$.
- $u(x, 0) = \sin(3\pi x)$.

4. **(a)** Use the technique of separation of variables to find a non-trivial solution of the differential equation,

$$u_{xx} + u_x = u_t$$

with the property that,

$$u(0, t) = 0 \text{ and } u(\pi, t) = 0 \text{ for all } t.$$

Note: There are many non-trivial solutions - you are required to find only one here.

(b) Find the limit of the solution you obtain as $t \to \infty$.

5. Use the technique of separation of variables to find solutions to the differential equations given below. Because no boundary or initil data is given, your solutions may contain some constants whose values are not specified.

(a) $u_x + u_y = 0.$

(b) $u_x - yu_y = 0.$

(c) $u_{xy} - u = 0.$

6. Find the temperature $u(x, t)$ in a bar of silver (length 10 cm) when the ends of the bar are kept at 0°C, the heat flow is governed by the equation,

$$u_t = 1.5752u_{xx}$$

and the initial temperature in the bar is given by the function,

$$f(x) = \begin{cases} x & ,0 < x < 5 \\ 10 - x & ,5 < x < 10 \end{cases}.$$

7. **(a)** Find the Fourier series of the following function,

$$f(x) = \begin{cases} 1 & ,-\pi/2 < x < \pi/2 \\ 0 & ,\pi/2 < x < 3\pi/2 \end{cases}.$$

(b) Using your result from part (a), show that:

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots = \tfrac{\pi}{4}.$$

8. In this problem, we will be dealing with a periodic function. The formula for one repetition of the function is given below.

$$f(x) = 1 - |x|\,, \quad -2 \le x \le 2.$$

(a) What is the period of this function?

(b) Find a series that uses sine and cosine functions to build a formula for the periodic function. If you are able to use geometrical insights to simplify some of your calculations, briefly describe the insight that helps you.

(c) Sketch a graph showing the periodic function and the fourth partial sum of your series from part (b). Where is the approximation the worst? (Remember that the formula $f(x) = 1 - |x|$ is only valid for $-2 \le x \le 2$.)

9. A second order, homogeneous, linear differential equation with constant coefficients is a differential equation of the form:

$$a\frac{d^2u}{dt^2} + b\frac{du}{dt} + cu = 0$$

where $u = u(t)$, and a, b and c are all constants.

(a) Show that $u(t) = Ke^{\lambda t}$ is a solution of this differential equation if and only if λ is a root of the characteristic equation.

(b) Suppose that the characteristic equation has two, purely imaginary roots $\pm i\omega$. Explain why both $\sin(\omega t)$ and $\cos(\omega t)$ are solutions of the differential equation.

10. In this problem the matrix A will always refer to the matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}.$$

(a) Calculate the eigenvalues of A.

(b) For each distinct eigenvalue of A, find a basis for the corresponding eigenspace.

(c) Suppose that a vector $\vec{x}$ satisfies the equation: $A\vec{x} = 7\cdot\vec{x}$. Calculate:

$$A^{-1}\vec{x}.$$

(Note: If it make this any easier, express your answer in terms of $\vec{x}$.)

(d) Suppose that B is an invertible n by n matrix and that $\lambda = 7$ is an eigenvalue of B. Suppose that $\vec{x}$ is an n dimensional vector that satisfies the equation: $B\vec{x} = 7\cdot\vec{x}$. Calculate $B^{-1}\vec{x}$.

11. In this problem, you will use the technique of "Least Squares" to fit various kinds of curves to a given set of data points. For all of the curves that you will attempt, the data points will be:

$$(-1, 0), (0, 1), (1, 3), (2, 5).$$

(a) The first kind of curve that you will fit to this data is a horizontal line of the form:

$$y = b$$

where b is a constant. Set up a system of equations and indicate why this system must be inconsistent.

(b) Set up the *normal equation* corresponding to the inconsistent system that you obtained in Part (a).

(c) Solve the normal equation from Part (b) and interpret your result as the equation for a horizontal line.

(d) Find the equation for the least squares best fit of a quadratic function:

$$y = ax^2 + bx + c$$

to the data points given.

12. Calculate the determinant of each of the following matrices:

(a) $A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 2 & -2 & 2 & 3 \\ 0 & 0 & 6 & 2 \end{bmatrix}.$

(b) $B = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$

(c) C is a 5 by 5 matrix that follows exactly the same pattern as B.

13. For each of the following statements, decide whether the statement is true or false.

(a) If A is an n by n matrix with n distinct (real) eigenvalues, then A is diagonalizable.

(b) If A is diagonalizable and $AB = BA$, then B is diagonalizable.

(c) If there is an invertible matrix P so that $A = P^{-1}BP$, then A and B have the same eigenvalues.

(d) If A and B have the same eigenvalues then there is an invertible matrix P so that $A = P^{-1}AP$.

(e) There is no real 2 by 2 matrix satisfying $A^2 = -I_2$.

(f) If A and B are diagonalizable and have the same eigenvalues (with the same algebraic multiplicities), then there is an invertible matrix P so that:

$$A = P^{-1}AP.$$

14. Decide whether each of the following statements is always true. If the statement is not always true, mark it false.

(a) A diagonizable matrix is invertible.

(b) Distinct eigenvectors of a real symmetric matrix are orthogonal.

(c) Suppose that a set of vectors is linearly independent. Then the set of vectors is also orthogonal.

(d) If the Gram-Schmidt process is used on a set of linearly independent vectors, then the process will never yield the zero vector.

(e) Let A be a real matrix, and λ be a complex eigenvalue of A. Then all eigenvectors of A belonging to λ will be complex.

15. Let A be an n by n diagonizable matrix and suppose that $P^{-1}AP = D$ where D is also a diagonalizable matrix. In such a case we could say that the two matrices A and D are *similar*.

(a) Show that A^T and D are similar.

(b) Show that there is an invertible matrix Q with the property that:

$$A^T = Q^{-1}AQ.$$

(c) Explain why A^T and A are similar.

16. Let A be a real, symmetric, non-zero matrix. Show that A cannot be nilpotent. (A matrix M is called nilpotent if there is some postive number n for which M^n is the zero matrix.)

17. In this problem, you may assume the areas bounded by the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is πab.

(a) If λ_1 and λ_2 are positive real numbers, show that the curve:

$$\lambda_1 x^2 + \lambda_2 y^2 = 1$$

is an ellipse bounding the area: $\pi/\sqrt{(\lambda_1\lambda_2)}$.

(b) If S is a 2 by 2 positive definite symmetric matrix, show that the equation:

$$\boldsymbol{x}^T S \boldsymbol{x} = 1$$

is an ellipse, and that this curve bounds an areas eqal to: $\pi/\sqrt{(\det(S))}$.

(c) Find the area of the region bounded by the ellipse:

$$5x^2 + 4xy + y^2 = 1.$$

18. Suppose that the 4 by 5 matrix A is the coefficient matrix of a homogeneous (i.e. equal to the zero vector) system of linear equations. The reduced row echelon form of A is given below.

$$\begin{bmatrix} 1 & -3 & 0 & -5 & 2 \\ 0 & 0 & 1 & 7 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find the solutions to the homogeneous system of equations. What is the dimension of the subspace of $\mathbf{R}^5$ that consists of all solutions to this system of equations ?

(b) Determine whether or not $\eta = (3, 2, 6, -1, -1)$ is a solution of the homogeneous system.

(c) A non-homeogeneous system that has the same coefficient matrix, A, as the homogeneous system has the solution $\rho_0 = (4, -1, 14, 6, -5)$. Is the vector:

$$\rho = (-3, 3, 15, -2, 1)$$

also a solution of the system?

19. Let A be a 4 by 4 matrix with the following properties:

(I) Two of the eigenvalues of A are: 3 and 2.

(II) 3 is an eigenvalue of the matrix $A + 2I_4$.

(III) $\det(A) = 12$.

(a) What are the other two eigenvalues of A ?

(b) What is the characteristic polynomial of A ?

(c) What is the characteristic polynomial of A^{-1} ?

20. The matrices given below represent the reduced row ecelon forms of the augmented matrices obtained from linear systems of equations, $Ax = b$. In each case, solve the system of linear equations.

(a) $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

21. Solve each of the following systems of linear equations.

(a) $5x_1 - 2x_2 + 6x_3 = 0$
$-2x_1 + x_2 + 3x_3 = 1$

(b) $x_1 - 2x_2 + x_3 - 4x_4 = 1$
$x_1 + 3x_2 + 7x_3 + 2x_4 = 2$
$x_1 - 12x_2 - 11x_3 - 16x_4 = 5$

(c) $w + 2x - y = 4$
$x - y = 3$
$w + 3x - 2y = 7$
$2u + 4v + w + 7x = 7$

22. Find the inverse of each of the following 4 by 4 matrices, where k_1, k_2, k_3, k_4 and k are all non-zero numbers.

(a) $\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$

23. Find diagonal matrices that satisfy the following two matrix equations. (Note: you don't have to come up with the same equation in both (a) and (b).)

(a) $A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **(b)** $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

24. Solve each of the following matrix equations to find X.

(a) $$X\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 5 \end{bmatrix}.$$

(b) $$X\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -1 & 0 \\ 6 & -3 & 7 \end{bmatrix}$$

(c) $$\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}X - X\begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 5 & 4 \end{bmatrix}$$

25. Use Cramer's Rule to solve for x' and y' in terms of x and y.

$$x = x' \cdot \cos(\theta) - y' \cdot \sin(\theta)$$

$$y = x' \cdot \sin(\theta) + y' \cdot \cos(\theta)$$

26. Recall that P_2 is the vector space of polynomial functions that have degree at most two. Determine whether each of the following sets of vectors from P_2 is linearly independent or not.

(a) $\{2 - x + 4x^2, 3 + 6x + 2x^2, 2 + 10x - 4x^2\}$

(b) $\{3 + x + x^2, 2 - x + 5x^2, 4 - 3x^2\}$

(c) $\{6 - x^2, 1 + x + 4x^2\}$

(d) $\{1 + 3x + 3x^2, x + 4x^2, 5 + 6x + 3x^2, 7 + 2x - x^2\}$

27. For which real values of λ do the following vectors form a linearly independent set in $\mathbf{R}^3$?

$$\vec{v}_1 = \begin{bmatrix} \lambda \\ \frac{-1}{2} \\ \frac{-1}{2} \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} \frac{-1}{2} \\ \lambda \\ \frac{-1}{2} \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ \lambda \end{bmatrix}$$

28. Recall that the set of all 2 by 2 matrices can be thought of as a vector space. Find the coordinates of the matrix A with respect to the basis $\{A_1, A_2, A_3, A_4\}$.

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

29. Each part of this problem describes a subspace of $\mathbf{R}^3$. In each case, find a basis for that subspace.

(a) The plane $3x - 2y + 5z = 0$.

(b) The plane $x - y = 0$.

(c) The line $x = 2t$, $y = -t$, $z = 4t$.

(d) All vectors of the form (a, b, c) where $b = a + c$.

30. In each part, find a basis for the subspace of P_2 that is spanned by the given set of vectors.

(a) $\{-1 + x - 2x^2, 3 + 3x + 6x^2, 9\}$

(b) $\{1 + x, x^2, -2 + 2x^2, -3x\}$

(c) $\{1 + x - 3x^2, 2 + 2x - 6x^2, 3 + 3x - 9x^2\}$

31. Find the rank and the dimension of the the kernel for each of the matrices given below.

(a) $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

32. Determine the rank of each of the matrices given below.

(a) $A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix}$.

Brief Answers

1.(a) 0.
1.(b) $\pi/54$.
1.(c) 0.
1.(d) 0.

2. $u(x, t) = (1/3)*\sin(3x)*\sin(6*t)$.

3. $u(x,t) = e^{\frac{-9\pi^2 (12)^2 t}{16}} \cdot \sin(3\pi x)$

4.(a) One of the many solutions is: $u(x,t) = e^{-x/2} \cdot \sin(x) \cdot e^{-5t/4}$.
4.(b) The limit as $t \to \infty$ is zero for every value of x.

5.(a) $u(x, y) = ke^{c(x - y)}$
5.(b) $u(x, y) = ky^c e^{cx}$
5.(c) $u(x, y) = ke^{cx + y/c}$.

6. $u(x,t) = \frac{40}{\pi^2}\left[e^{-0.01752\pi^2 t} \cdot \sin\left(\frac{\pi}{10}x\right) - \frac{1}{9}e^{-0.01752(3\pi)^2 t} \cdot \sin\left(\frac{3\pi}{10}x\right) + \ldots\right]$

7.(a) $\frac{1}{2} + \frac{2}{\pi}\left[\cos(x) - \frac{1}{3}\cos(3x) + \frac{1}{5}\cos(5x) - \ldots\right]$
7.(b) Substitute $x = 0$ into the series from part (a), and into $f(x)$.

8.(a) Period = 4.
8.(b) $\sum_{n=1}^{\infty} \frac{4\left(1-(-1)^n\right)}{n^2\pi^2} \cos\left(\frac{n\pi}{2}x\right)$.
8.(c) Whenever x is an integer multiple of 2 the Fourier approximation is not so hot. This is because the graph of $y = f(x)$ is not differentiable whenever x is an integer multiple of 2.

9.(a) Substitute $u(t)$ into the differential equation and simplify as much as you can. You will be left with the condition that the characteristic equation with λ substituted into it must be zero in order for the differential equation to be satisfied.
9.(b) From (a) the solutions are $u(t) = Ae^{\pm\omega it}$. Use Euler's formula to express $\sin(\omega t)$ and $\cos(\omega t)$ in terms of $e^{\pm\omega it}$. The final step is to equate real and imaginary parts, and to conclude that the real part and the imaginary parts must be independent solutions of the differential equation.

10.(a) The eigenvalues of A are 1 and 7.

10.(b) A basis for the eigenspace of $\lambda = 1$ is given by: $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$.

A basis for the eigenspace of $\lambda = 7$ is given by: $\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\}$.

10.(c) $A^{-1}x = (1/7)*x$.
10.(d) $B^{-1}x = (1/7)*x$.

11.(c) $y = 9/4$.
11.(d) $y = (1/4)x^2 + (29/20)x + (23/20)$.

12.(a) $\det(A) = -4$.
12.(b) $\det(B) = 5$.
12.(c) $\det(C) = 6$.

13.(a) True.
13.(b) False. For example, could have $A = n$ by n zero matrix and $B =$ any other n by n matrix.
13.(c) True.
13.(d) False. For example, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
13.(e) False. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.
13.(f) True. Have $A = R^{-1}DR$ and $B = Q^{-1}DQ$. Take $P = R^{-1}Q$.

14.(a) False.
14.(b) False.
14.(c) False.
14.(d) True.
14.(e) False.

15.(a) $A = PDP^{-1}$. Since D is diagonizable, there is an invertible matrix B and a diagonal matrix C so that: $D = BCB^{-1}$. Then: $A^T = ((PB)^T)^{-1}C(PB)^T = R^{-1}DR$, where $R = B(PB)^T$.
15.(b) Since A is diagonizable, there is an invertible matrix M and a diagonal matrix W so that $A = MWM^{-1}$. Then, $A^T = (M^T)^{-1}WM^T = (M^T)^{-1}M^{-1}MWM^{-1}MM^T$. Set $Q = MM^T$. Then: $A^T = Q^{-1}WQ$.
15.(c) This is implied by the result of (b).

16. If A is real and symmtric, then it can be diagonalized with $A = PDP^{-1}$. If A is non-zero, then not all entries of the diagonal matrix, D, are zero. Recall that to raise a diagonal matrix to a power, you raise the diagonal entries to the power. Hence, no power of D is the zero matrix. Since:$A^n = PD^nP^{-1}$, A^n cannot be zero either.

17.(c) Area = π.

18.(a) The subspace is the span of the three vectors: (3, 1, 0, 0, 0), (5, 0, -7, 1, 0), (-2, 0, 1, 0, 1).
18.(b) Yes.
18.(c) No.

19.(a) The other two eigenvalues are 1 and 2.
19.(b) Characteristic polynomial = $(x - 1)(x - 2)^2(x - 3)$.
19.(c) The eigenvalues of A^{-1} are 1, 1/2, 1/2 and 1/3. The characteristic polynomial of A^{-1} is: $(x - 1)(x - 1/2)^2(x - 1/3)$.

20.(a) $x_1 = -3$, $x_2 = 0$ and $x_3 = 7$.
20.(b) This system is inconsistent and has no solutions.

21.(a) $x_1 = 2 - 12x_3$, $x_2 = 5 - 27x_3$, and $x_3 = x_3$.
21.(b) This system is inconsistent.
21.(c) $u = -2v - 3y - 6$, $v = v$, $w = -y - 2$, $x = y + 3$, $y = y$.

22.(a) $\begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & \frac{1}{k_4} \end{bmatrix}$.

22.(b) $\begin{bmatrix} 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ \frac{1}{k_1} & 0 & 0 & 0 \end{bmatrix}$.

22.(c) $\begin{bmatrix} \frac{1}{k} & 0 & 0 & 0 \\ \frac{-1}{k^2} & \frac{1}{k} & 0 & 0 \\ \frac{1}{k^3} & \frac{-1}{k^2} & \frac{1}{k} & 0 \\ \frac{-1}{k^4} & \frac{1}{k^3} & \frac{-1}{k^2} & \frac{1}{k} \end{bmatrix}$.

23.(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

23.(b) $A = \begin{bmatrix} \pm\frac{1}{3} & 0 & 0 \\ 0 & \pm\frac{1}{2} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$.

24.(a) $X = \begin{bmatrix} -1 & 3 & -1 \\ 6 & 0 & 1 \end{bmatrix}$

24.(b) $X = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$

24.(c) $X = \begin{bmatrix} \frac{-113}{37} & \frac{-160}{37} \\ \frac{-20}{37} & \frac{-46}{37} \end{bmatrix}$

25.
$$x' = x \cdot \cos(\theta) + y \cdot \sin(\theta)$$
$$y' = -x \cdot \sin(\theta) + y \cdot \cos(\theta)$$

26.(a) Not linearly independent.
26.(b) Not linearly independent.
26.(c) Not linearly independent.
26.(d) Linearly independent.

27. $\lambda = -1/2$, $\lambda = 1$.

28. Keeping the order that the "basis matrices" were presented in the question in mind, the coordinates are: -1, 1, -1, 3.

29.(a) $\left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-5}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$

29.(b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

29.(c) $\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right\}$

29.(d) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

30.(a) $\{-1 + x - 2x^2, 3 + 3x + 6x^2, 9\}$
30.(b) $\{1 + x, x^2, -2 + 2x^2\}$
30.(c) $\{1 + x - 3x^2\}$

31.(a) dim(ker(A)) = 1, rank(A) = 2.
31.(b) dim(ker(A)) = 2, rank(A) = 1.
31.(c) dim(ker(A)) = 2, rank(A) = 2.

32.(a) $rank(A) = \begin{cases} 1 & ,t = 1 \\ 2 & ,t = -2 \\ 3 & ,t \neq 1, -2 \end{cases}$.

32(b) $rank(A) = \begin{cases} 2 & ,t = 1, \frac{3}{2} \\ 3 & ,t \neq 1, \frac{3}{2} \end{cases}$