Practice Problems: Final Exam

Please note: *I have chosen these problems because I think that they are representative of many of the mathematical concepts that we have studied. There is no guarantee that the problems that appear on any test will resemble these problems in any way whatsoever.*

Remember: *On exams, you have to supply evidence for your conclusions, and explain why your answers are appropriate.*

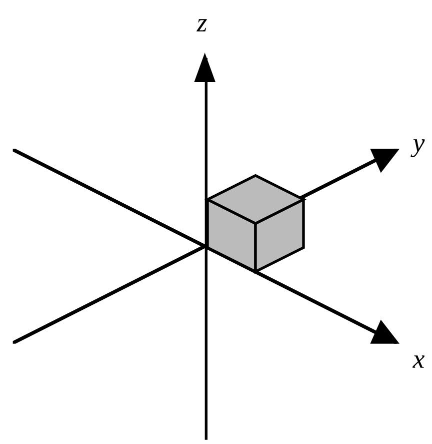
1. Calculate the Fourier series expansions for each of the following functions.

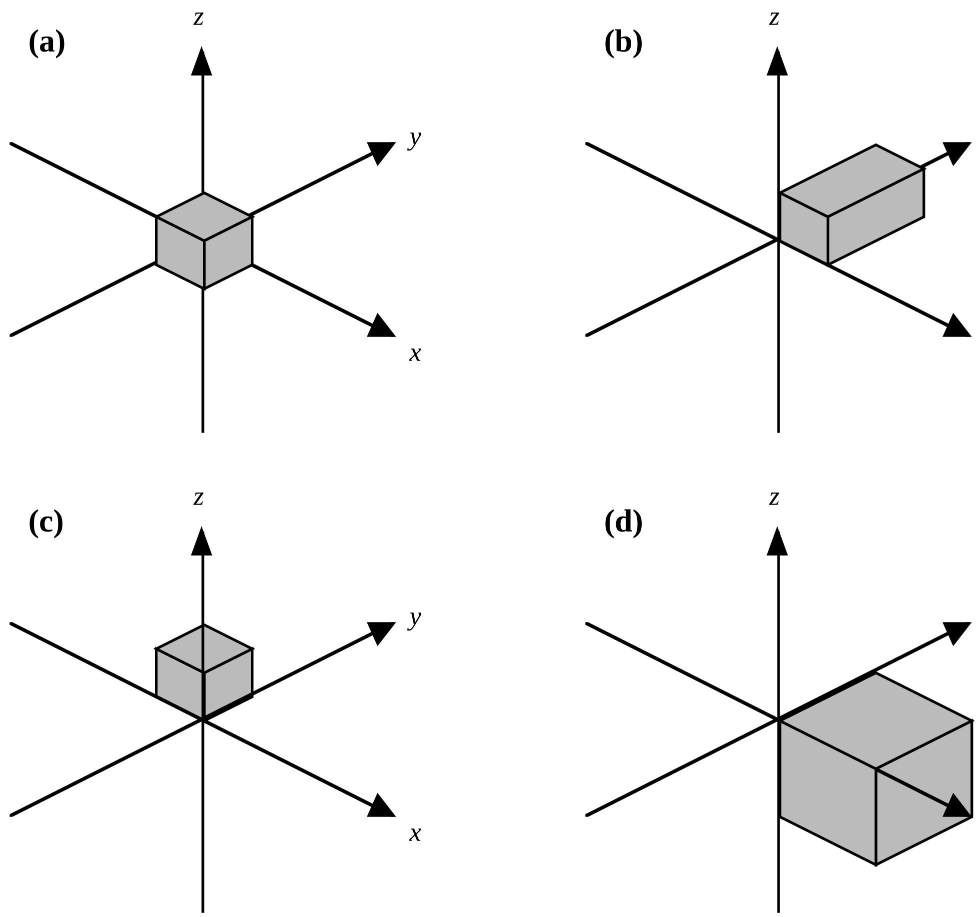
(a) $$f(x) = \begin{cases} x & , -\pi \le x < 0 \\ 0 & , 0 \le x < \pi \end{cases}$$

(b) $$f(x) = \begin{cases} x + L & , -L \le x < 0 \\ L & , 0 \le x < L \end{cases}$$ where L is a positive real number.

(c) $$f(x) = \begin{cases} -1 & , -2 \le x < 0 \\ 1 & , 0 \le x < 2 \end{cases}$$

2. The diagram shown below depicts a cube whose sides line up with the coordinate axes in three dimensional space. Each side of this cube has a length of 1. Each of the diagrams given below that was obtained from this prototypical cube by means of a transformation $T{:}\mathbf{R}^3 \to \mathbf{R}^3$. In each case decide whether or not the transformation was a linear transformation. If you believe that the transformation was linear, find a matrix to represent the transformation.





3. Recall that the symbol P_2 represents the vector space consisting of all polynomials with degree of less than or equal to 2.

(a) Show that the three of polynomials:

$$p_1(x) = 1 + x$$

$$p_2(x) = 1 + x^2$$

$$p_3(x) = x + x^2$$

form a basis for P_2.

(b) Express the polynomial function: $p(x) = 2 - x + x^2$ as a linear combination of $p_1(x)$, $p_2(x)$ and $p_3(x)$.

4. Recall that the symbol P_2 represents the vector space of polynomial functions with degree of less than or equal to 2. The collection of functions:

$$f_1(x) = 1$$

$$f_2(x) = x$$

$$f_3(x) = x^2$$

forms a basis of P_2 called the "Standard Basis." It is possible to define an inner product on P_2 that behaves in much the same way as the "dot" product of $\mathbf{R}^n$. This inner product is defined by the equation:

$$p(x) \bullet q(x) = \int_{-1}^{1} p(x) \cdot q(x) dx .$$

Apply the Gram-Schmidt process to the standard basis to obtain an orthonormal basis for P_2.

5. In this problem, A will always refer to the three by three matrix defined below.

$$A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Calculate the characteristic polynomial of A.

(b) Find the eigenvalues of A and their algebraic multiplicities.

(c) For each (distinct) eigenvalue that you found on Part (b), calculate a basis for the eigenspace.

(d) Determine the geometric multiplicity of each eigenvalue found in Part (b).

(e) Is A a diagonalizable matrix? If so, find an invertible three by three matrix P such that $P^{-1}AP$ is a diagonal matrix.

(f) Calculate A^{2301}.

6. The point of this problem is to find a non-trivial function $u(x, t)$ -- that is, a function that is not simply equal to zero all of the time -- that satisfies all of the conditions given below.

- $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ when $0 \le x \le 40$ and $t > 0$.
- $u(0, t) = u(40, t) = 0$ when $t > 0$.
- $u(x,0) = \begin{cases} 0 & ,0 \le x < 10 \\ 50 & ,10 \le x \le 30 \\ 0 & ,30 < x \le 40 \end{cases}$ when $0 \le x \le 40$

7. In each of the following cases, find the solution of the ordinary differential equation that satisfies the given conditions.

(a) $2y'' + y' - 4y = 0$ $\quad y(0) = 0$ $\quad y'(0) = 1$

(b) $y'' + 8y' - 9y = 0$ $\quad y(1) = 1$ $\quad y'(1) = 0$

(c) $4y'' - y = 0$ $\quad y(-2) = 1$ $\quad y'(-2) = -1$

8. In this problem, A will always refer to the two by two matrix given below.

$$A = \begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}.$$

(a) Find the characteristic polynomial of A.

(b) Calculate the eigenvalues of A and find bases for the corresponding eigenspaces.

(c) Consider the discrete dynamical system:

$$\begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = A \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \text{ with initial condition: } \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Find explicit, easy-to-evaluate formulas for the functions $x(t)$ and $y(t)$.

Brief Answers

1.(a) $$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{2\cdot\cos[(2n-1)x]}{\pi(2n-1)^2} + \frac{(-1)^{n+1}\cdot\sin(nx)}{n}\right]$$

1.(b) $$f(x) = \frac{3L}{4} + \sum_{n=1}^{\infty}\left[\frac{2L\cdot\cos[(2n-1)\pi x/L]}{\pi^2\cdot(2n-1)^2} + \frac{(-1)^{n+1}\cdot L\cdot\sin(n\pi x/L)}{n\pi}\right]$$

1.(c) $$f(x) = \frac{4}{\pi}\cdot\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi x/2]}{2n-1}$$

2.(a) Linear. Matrix representation: $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2.(b) Linear. Matrix representation: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2.(c) Linear. Matrix representation: $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2.(d) Linear. Matrix representation: $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix}$.

3.(a) The vector space P_2 is three dimensional, so it is sufficient to show that the set of three vectors is linearly independent, that is, to show that the linear equation:

$$c_1\cdot p_1(x) + c_2\cdot p_2(x) + c_3\cdot p_3(x) = 0$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$. The above linear equation is equivalent to the system of linear equations:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the determinant of the coefficient matrix is non-zero, this system has only the trivial solution $c_1 = c_2 = c_3 = 0$.

3.(b) $p(x) = 2\cdot p_2(x) - p_3(x)$.

4. The three members of the orthonormal basis are:

$$w_1(x) = \frac{1}{\sqrt{2}} \qquad w_2(x) = \sqrt{\frac{3}{2}}x \qquad w_3(x) = \frac{3\sqrt{5}}{\sqrt{33-10\sqrt{6}}}\left(x^2 - \frac{1}{\sqrt{6}}\right).$$

5.(a) The characteristic polynomial of A is: $f_A(x) = (x-1)(x+1)^2$.

5.(b) The eigenvalues of A are: 1 (algebraic multiplicity = 1) and –1 (algebraic multiplicity = 2).

5.(c) Basis for eigenspace of $\lambda = 1$: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$. Basis for eigenspace of $\lambda = -1$: $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-4\\0\\1\end{bmatrix}\right\}$.

5.(d) The geometric multiplicity of $\lambda = 1$ is equal to 1. The geometric multiplicity of $\lambda = -1$ is equal to 2.

5.(e) Yes A is diagonalizable, as the geometric multiplicities add up to 3. A matrix that does the required job is given by:

$$P = \begin{bmatrix}1 & 1 & -4\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

5.(f) $A^{2301} = A$.

6. $$u(x,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{3n\pi}{4}\right)}{n} e^{-n^2\pi^2 t/1600}\cdot\sin\left(\frac{n\pi x}{40}\right)$$

7.(a) $$y = \frac{2}{\sqrt{33}}e^{\left(-1+\sqrt{33}\right)t/4} - \frac{2}{\sqrt{33}}e^{\left(-1-\sqrt{33}\right)t/4}$$

7.(b) $$y = \frac{1}{10}e^{-9(t-1)} + \frac{9}{10}e^{t-1}$$

7.(c) $$y = -\frac{1}{2}e^{(t+2)/2} + \frac{3}{2}e^{-(t+2)/2}$$

8.(a) The characteristic polynomial of A is: $f_A(x) = (x-11)(x+7)$.

8.(b) Eigenvalue: $2 + 3I$. Basis of eigenspace: $\begin{bmatrix}1.5+0.5i\\1\end{bmatrix}$.

Eigenvalue: $2 - 3I$. Basis of eigenspace: $\begin{bmatrix}1.5-0.5i\\1\end{bmatrix}$.

8.(c) Let $\varphi = \tan^{-1}(1.5) \approx 0.9828$. Then: $x(t) = \cos(\varphi t) + 3\cdot\sin(\varphi t)$ and $y(t) = \sin(\varphi t)$.