Section 3.1

25. $\text{im}(T) = \mathbb{R}^2$ and $\ker(T) = \{\vec{0}\}$, since T is invertible (see Summary 3.1.8).

37. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so that
$\ker(A) = \text{span}(\vec{e}_1)$, $\ker(A^2) = \text{span}(\vec{e}_1, \vec{e}_2)$, $\ker(A^3) = \mathbb{R}^3$, and
$\text{im}(A) = \text{span}(\vec{e}_1, \vec{e}_2)$, $\text{im}(A^2) = \text{span}(\vec{e}_1)$, $\text{im}(A^3) = \{\vec{0}\}$.

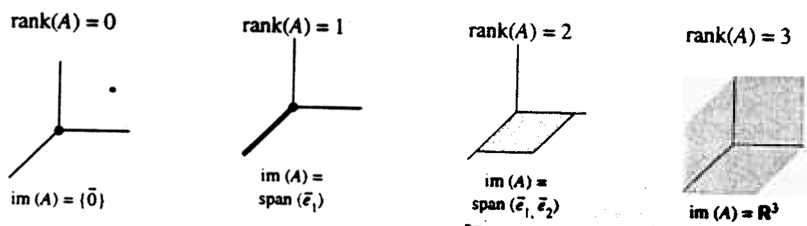
38. a. If a vector $\vec{x}$ is in $\ker(A^k)$, that is, $A^k\vec{x} = \vec{0}$, then $\vec{x}$ is also in $\ker(A^{k+1})$, since $A^{k+1}\vec{x} = AA^k\vec{x} = A\vec{0} = \vec{0}$.
Therefore, $\ker(A) \subseteq \ker(A^2) \subseteq \ker(A^3) \ldots$
Exercise 37 shows that these kernels need not be equal.

b. If a vector $\vec{y}$ is in $\text{im}(A^{k+1})$, that is, $\vec{y} = A^{k+1}\vec{x}$ for some $\vec{x}$, then $\vec{y}$ is also in $\text{im}(A^k)$, since we can write $\vec{y} = A^k(A\vec{x})$. Therefore, $\ldots \text{im}(A^3) \subseteq \text{im}(A^2) \subseteq \text{im}(A)$.
Exercise 37 shows that these images need not be equal.

44. a. Yes; by construction of the echelon form, the systems $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solutions (it is the whole point of Gaussian elimination not to change the solutions of a system).

b. No; as a counterexample, consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, with $\text{im}(A) = \text{span}(\vec{e}_2)$, but $B = \text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, with $\text{im}(B) = \text{span}(\vec{e}_1)$.

46. If $\text{rank}(A) = r$, then $\text{im}(A) = \text{span}(\vec{e}_1, \ldots, \vec{e}_r)$.



51. We need to find all $\vec{x}$ such that $AB\vec{x} = \vec{0}$. If $AB\vec{x} = \vec{0}$, then $B\vec{x}$ is in $\ker(A) = \{\vec{0}\}$, so that $B\vec{x} = \vec{0}$. Since $\ker(B) = \{\vec{0}\}$, we can conclude that $\vec{x} = \vec{0}$. It follows that $\ker(AB) = \{\vec{0}\}$.

Section 3.2

6. a. Yes!

- The zero vector is in $V \cap W$, since $\vec{0}$ is in both V and W.
- If $\vec{x}$ and $\vec{y}$ are in $V \cap W$, then both $\vec{x}$ and $\vec{y}$ are in V, so that $\vec{x} + \vec{y}$ is in V as well, since V is a subspace of $\mathbb{R}^n$. Likewise, $\vec{x} + \vec{y}$ is in W, so that $\vec{x} + \vec{y}$ is in $V \cap W$.
- If $\vec{x}$ is in $V \cap W$ and k is an arbitrary scalar, then $k\vec{x}$ is in both V and W, since they are subspaces of $\mathbb{R}^n$. Therefore, $k\vec{x}$ is in $V \cap W$.

b. No; as a counterexample consider $V = \text{span}(\vec{e}_1)$ and $W = \text{span}(\vec{e}_2)$ in $\mathbb{R}^2$.

7. Yes; we need to show that W contains the zero vector. We are told that W is nonempty, so that it contains some vector $\vec{v}$. Since W is closed under scalar multiplication, it will contain the vector $0\vec{v} = \vec{0}$, as claimed.

22. If a, c and f are nonzero, then $\text{rref}\begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and the three vectors are linearly independent, by Fact 3.2.6. If at least one of the constants a, c or f is zero, then at least one column of rref will not contain a leading one, so that the three vectors are linearly dependent.

28. The three column vectors are linearly independent, since $\text{rref}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix} = I_3$.

Therefore, the three columns form a basis of $\text{im}(A)(= \mathbb{R}^3)$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$$

Another sensible choice for a basis of $\text{im}(A)$ is $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

35. Consider a nontrival relation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \vec{0}$.
Let i be the highest index such that $c_i \neq 0$: since $\vec{v}_1 \neq \vec{0}$ we know that $i > 1$. Now we have $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_i\vec{v}_i = \vec{0}$.
We can solve this relation for $\vec{v}_i$ and thus express $\vec{v}_i$ as a linear combination $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}$:
$\vec{v}_i = -\frac{c_1}{c_i}\vec{v}_1 - \frac{c_2}{c_i}\vec{v}_2 - \cdots - \frac{c_{i-1}}{c_i}\vec{v}_{i-1}$.

37. No; as a counterexample, consider the extreme case when T is the zero transformation, that is, $T(\vec{x}) = \vec{0}$ for all $\vec{x}$. Then the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_m)$ will all be zero, so that they are linearly dependent.

Section 3.3

8. $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & -2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

A basis of $\ker(A)$ is $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, so that $\dim(\ker(A)) = 3$.

22. $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

A basis of $\text{im}(A)$ is $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$, so that $\dim(\text{im}(A)) = 3$.

A basis of $\ker(A)$ is $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, so that $\dim(\ker(A)) = 2$.

$\dim(\text{im}(A)) + \dim(\ker(A)) = 5 = \#$ of columns, in accordance with Fact 3.3.9.

$\ldots \text{rank}(A) \geq n - m$, by Fact 3.3.5.

35. We need to find all vectors $\vec{x}$ in $\mathbb{R}^n$ such that $\vec{v} \cdot \vec{x} = 0$, or $v_1x_1 + v_2x_2 + \cdots + v_nx_n = 0$, where the v_i are the components of the vector $\vec{v}$. These vectors form a hyperplane in $\mathbb{R}^n$ (see Exercise 33), so that the dimension of the space is $n - 1$.

36. No; if $\text{im}(A) = \ker(A)$ for an $n \times n$ matrix A, then $n = \dim(\ker(A)) + \dim(\text{im}(A)) = 2\dim(\text{im}(A))$, so that n is an even number.

37. Since $\dim(\ker(A)) = 5 - \text{rank}(A)$, any 4×5 matrix with rank 2 will do; for example,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

38. a. The rank of a 3×5 matrix A is 0,1,2, or 3, so that $\dim(\ker(A)) = 5 - \text{rank}(A)$ is 2,3,4, or 5.

b. The rank of a 7×4 matrix A is at most 4, so that $\dim(\text{im}(A)) = \text{rank}(A)$ is 0,1,2,3, or 4.