

3. This subset  $V$  is a subspace of  $P_2$ :

- The neutral element  $f(t) = 0$  (for all  $t$ ) is in  $V$  since  $f'(1) = f(2) = 0$ .
- If  $f$  and  $g$  are in  $V$  (so that  $f'(1) = f(2)$  and  $g'(1) = g(2)$ ), then  $(f+g)'(1) = (f'+g')(1) = f'(1) + g'(1) = f(2) + g(2) = (f+g)(2)$ , so that  $f+g$  is in  $V$ .
- If  $f$  is in  $V$  (so that  $f'(1) = f(2)$ ) and  $k$  is any constant, then  $(kf)'(1) = (kf')(1) = kf'(1) = kf(2) = (kf)(2)$ , so that  $kf$  is in  $V$ .

If  $f(t) = a + bt + ct^2$  then  $f'(t) = b + 2ct$ , and  $f$  is in  $V$  if  $f'(1) = b + 2c = a + 2b + 4c = f(2)$ , or  $a + b + 2c = 0$ . The general element of  $V$  is of the form  $f(t) = (-b - 2c) + bt + ct^2 = b(t - 1) + c(t^2 - 2)$ , so that  $t - 1, t^2 - 2$  is a basis of  $V$ .

8. This is a subspace; the justification is analogous to Exercise 7.

1. Let  $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ . Let  $V$  be the set of all  $3 \times 3$  matrices  $A$  such that  $A\vec{v} = \vec{0}$ . Then  $V$  is a subspace of  $\mathbb{R}^{3 \times 3}$ :

- The zero matrix  $0$  is in  $V$ , since  $0\vec{v} = \vec{0}$ .
- If  $A$  and  $B$  are in  $V$ , then so is  $A + B$ , since  $(A + B)\vec{v} = A\vec{v} + B\vec{v} = \vec{0} + \vec{0} = \vec{0}$ .
- If  $A$  is in  $V$ , then so is  $kA$  for all scalars  $k$ , since  $(kA)\vec{v} = k(A\vec{v}) = k\vec{0} = \vec{0}$ .

20. Use the strategy outlined on Page 155. We have  $a = -d$ , so that the general element of the subspace

$$\text{is } \begin{bmatrix} -d & b \\ c & d \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$  is a basis of the subspace; the dimension is 3.

29. We are looking for the matrices  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \text{ It is required that } a = -b \text{ and } c = -d. \text{ Thus the general}$$

element is  $\begin{bmatrix} -b & b \\ -d & d \end{bmatrix} = b \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$ . Thus  $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$  is a basis, and the dimension is 2.

31. We are looking for the matrices  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ or, } \begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}. \text{ It is required that } a = c \text{ and } b = -d.$$

Thus the general element is  $\begin{bmatrix} c & -d \\ c & d \end{bmatrix} = c \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$ . Thus

$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$  is a basis, and the dimension is 2.

#### 4.2

1. Fails to be linear, since  $T(A+B) = A+B+I_2$  doesn't equal  $T(A)+T(B) = A+I_2+B+I_2 = A+B+2I_2$ .

2. Linear, since  $T(A+B) = 7(A+B) = 7A+7B$  equals  $T(A)+T(B) = 7A+7B$ , and  $T(kA) = 7kA$  equals  $kT(A) = k(7A) = 7kA$ .  
Yes,  $T$  is an isomorphism, with  $T^{-1}(A) = \frac{1}{7}A$ .

3. Linear, since  $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) = T\begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix} = a+p+d+s$

$$\text{equals } T\begin{bmatrix} a & b \\ c & d \end{bmatrix} + T\begin{bmatrix} p & q \\ r & s \end{bmatrix} = a+d+p+s, \text{ and } T\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = ka+kd$$

$$\text{equals } kT\begin{bmatrix} a & b \\ c & d \end{bmatrix} = k(a+d) = ka+kd.$$

No,  $T$  fails to be an isomorphism, since  $4 = \dim(\mathbb{R}^{2 \times 2}) \neq \dim(\mathbb{R}) = 1$ ; see Fact 4.2.3d.

4. Fails to be linear, since  $T(2I_2) = \det(2I_2) = 4$  does not equal  $2T(I_2) = 2\det(I_2) = 2$ .

5. Fails to be linear, since  $T(2I_2) = 4I_2$  does not equal  $2T(I_2) = 2I_2$ .

22. Linear, since  $T(f(t)+g(t)) = \begin{bmatrix} f(7)+g(7) \\ f(11)+g(11) \end{bmatrix}$  equals  $T(f(t))+T(g(t)) =$

$$\begin{bmatrix} f(7) \\ f(11) \end{bmatrix} + \begin{bmatrix} g(7) \\ g(11) \end{bmatrix} = \begin{bmatrix} f(7)+g(7) \\ f(11)+g(11) \end{bmatrix}, \text{ and } T(kf(t)) = \begin{bmatrix} kf(7) \\ kf(11) \end{bmatrix} \text{ equals}$$

$$kT(f(t)) = k\begin{bmatrix} f(7) \\ f(11) \end{bmatrix} = \begin{bmatrix} kf(7) \\ kf(11) \end{bmatrix}.$$

Not an isomorphism, since domain and codomain have different dimensions.

26. We need to find the matrices  $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  such that

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x + 2z & y + 2t \\ 3x + 6z & 3y + 6t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \text{ It is required that } x = -2z \text{ and}$$

$y = -2t$ , so that the kernel of  $T$  consists of all matrices of the form  $\begin{bmatrix} -2z & -2t \\ z & t \end{bmatrix}$

The nullity (i.e., the dimension of the kernel of  $T$ ) is 2.

34. Note that  $T(a + bt + ct^2) = \begin{bmatrix} a + 7b + 49c \\ a + 11b + 121c \end{bmatrix}$ . To find the kernel, solve the linear

system  $\begin{bmatrix} a + 7b + 49c \\ a + 11b + 121c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . The solution is  $a = 77c$ ,  $b = -18c$ , so that the

kernel consists of all polynomials of the form

$f(t) = c(77 - 18t + t^2) = c(t - 11)(t - 7)$ . You can also see directly that the quadratic polynomials  $f(t)$  with  $f(7) = f(11) = 0$  are of this form. The nullity is 1.

The image consists of all of  $\mathbb{R}^2$ , so that the rank is 2.