11. a. $\alpha_n = \arccos \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \arccos \frac{1}{\sqrt{n}}$

$\alpha_2 = \arccos \frac{1}{\sqrt{2}} = \frac{\pi}{4} (= 45^\circ)$

$\alpha_3 = \arccos \frac{1}{\sqrt{3}} \approx 0.955$ (radians)

$\alpha_4 = \arccos \frac{1}{2} = \frac{\pi}{3} (= 60^\circ)$

b. Since $y = \arccos(x)$ is a continuous function, $\lim_{n\to\infty} \alpha_n = \arccos\left(\lim_{n\to\infty} \frac{1}{\sqrt{n}}\right) = \arccos(0) = \frac{\pi}{2} (= 90^\circ)$

16. You may be able to find the solutions by educated guessing. Here is the systematic approach: we first find all vectors $\vec{x}$ that are orthogonal to $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, then we identify the unit vectors among them. Finding the vectors $\vec{x}$ with $\vec{x}\cdot\vec{v}_1 = \vec{x}\cdot\vec{v}_2 = \vec{x}\cdot\vec{v}_3 = 0$ amounts to solving the system

$$\left| \begin{array}{l} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{array} \right|$$

(we can omit all the coefficients $\frac{1}{2}$).

The solutions are of the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ -t \\ t \end{bmatrix}$.

Since $\|\vec{x}\| = 2|t|$, we have a unit vector if $t = \frac{1}{2}$ or $t = -\frac{1}{2}$. Thus there are two possible choices for $\vec{v}_4$:

$\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

22. Let $W = \{\vec{x} \text{ in } \mathbb{R}^n : \vec{x} \cdot \vec{v}_i = 0 \text{ for all } i = 1, \ldots, m\}$. We are asked to show that $V^\perp = W$, that is, any $\vec{x}$ in $V^\perp$ is in W, and vice versa.
If $\vec{x}$ is in $V^\perp$, then $\vec{x} \cdot \vec{v} = 0$ for all $\vec{v}$ in V; in particular, $x \cdot \vec{v}_i = 0$ for all i (since the $\vec{v}_i$ are in V), so that $\vec{x}$ is in W.

Conversely, consider a vector $\vec{x}$ in W. To show that $\vec{x}$ is in $V^\perp$, we have to verify that $\vec{x} \cdot \vec{v} = 0$ for all $\vec{v}$ in V. Pick a particular $\vec{v}$ in V. Since the $\vec{v}_i$ span V, we can write $\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$, for some scalars c_i. Then $\vec{x} \cdot \vec{v} = c_1(\vec{x} \cdot \vec{v}_1) + \cdots + c_m(\vec{x} \cdot \vec{v}_m) = 0$, as claimed.

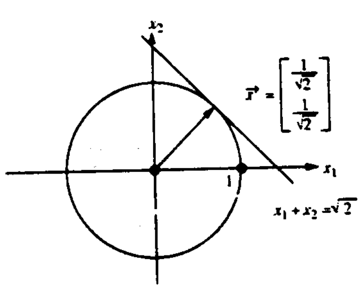
24. Write $T(\vec{x}) = \text{proj}_V(\vec{x})$ for simplicity.
To prove the linearity of T we will use the definition of a projection: $T(\vec{x})$ is in V, and $\vec{x} - T(\vec{x})$ is in $V^\perp$.
To show that $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$, note that $T(\vec{x}) + T(\vec{y})$ is in V (since V is a subspace), and $\vec{x} + \vec{y} - (T(\vec{x}) + T(\vec{y})) = (\vec{x} - T(\vec{x})) + (\vec{y} - T(\vec{y}))$ is in $V^\perp$ (since $V^\perp$ is a subspace, by Fact 5.1.5).
To show that $T(k\vec{x}) = kT(\vec{x})$, note that $kT(\vec{x})$ is in V (since V is a subspace), and $k\vec{x} - kT(\vec{x}) = k(\vec{x} - T(\vec{x}))$ is in $V^\perp$ (since $V^\perp$ is a subspace).

28. Since the three given vectors in the subspace are orthogonal, we have the orthonormal basis

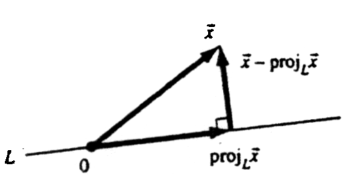
$$\vec{v}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Now we can use Fact 5.1.6, with $\vec{x} = \vec{e}_1$: $\text{proj}_V\vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + (\vec{v}_2 \cdot \vec{x})\vec{v}_2 + (\vec{v}_3 \cdot \vec{x})\vec{v}_3 = \frac{1}{4}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

34. Let $\vec{x}$ be a unit vector in $\mathbb{R}^n$, that is, $\|\vec{x}\| = 1$. Let $\vec{y} = \begin{bmatrix} 1 \\ \cdots \\ 1 \end{bmatrix}$ (all n components are 1). The Cauchy-Schwarz inequality (Fact 5.1.10) tells us that $|\vec{x} \cdot \vec{y}| \le \|\vec{x}\|\|\vec{y}\|$, or, $|x_1 + \ldots + x_n| \le \|\vec{x}\|\sqrt{n} = \sqrt{n}$. By Fact 5.1.10, the equation $x_1 + \ldots + x_n = \sqrt{n}$ holds if $\vec{x} = k\vec{y}$ for positive k. Thus $\vec{x}$ must be a unit vector of the form $\vec{x} = \begin{bmatrix} k \\ \cdots \\ k \end{bmatrix}$ for some positive k. It is required that $nk^2 = 1$, or, $k = \frac{1}{\sqrt{n}}$. Thus $\vec{x} = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \cdots \\ \frac{1}{\sqrt{n}} \end{bmatrix}$ $\left(\text{all components are } \frac{1}{\sqrt{n}}\right)$.
The figure below illustrates the case $n = 2$.



39. No! By definition of a projection, the vector $\vec{x} - \text{proj}_L\vec{x}$ is perpendicular to $\text{proj}_L\vec{x}$, so that $(\vec{x} - \text{proj}_L\vec{x}) \cdot (\text{proj}_L\vec{x}) = \vec{x} \cdot \text{proj}_L\vec{x} - \|\text{proj}_L\vec{x}\|^2 = 0$ and $\vec{x} \cdot \text{proj}_L\vec{x} = \|\text{proj}_L\vec{x}\|^2 \ge 0$.



Section 5.2

4. $\vec{w}_1 = \frac{1}{5}\begin{bmatrix}4\\0\\3\end{bmatrix}$ and $\vec{w}_2 = \frac{1}{5}\begin{bmatrix}3\\0\\-4\end{bmatrix}$ as in Exercise 3.

Since $\vec{v}_3$ is orthogonal to $\vec{w}_1$ and $\vec{w}_2$, $\vec{w}_3 = \frac{1}{\|\vec{v}_3\|}\vec{v}_3 = \begin{bmatrix}0\\-1\\0\end{bmatrix}$

8. $\vec{w}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{7}\begin{bmatrix}5\\4\\2\\2\end{bmatrix}$

$\vec{w}_2 = \frac{\vec{v}_2 - (\vec{w}_1\cdot\vec{v}_2)\vec{w}_1}{\text{length}} = \frac{1}{7}\begin{bmatrix}-2\\2\\5\\-4\end{bmatrix}$

18. $Q = \frac{1}{5}\begin{bmatrix}4 & 3 & 0\\0 & 0 & -5\\3 & -4 & 0\end{bmatrix}$, $R = \begin{bmatrix}5 & 5 & 0\\0 & 35 & 0\\0 & 0 & 2\end{bmatrix}$

22. $Q = \frac{1}{7}\begin{bmatrix}5 & -2\\4 & 2\\2 & 5\\2 & -4\end{bmatrix}$, $R = \begin{bmatrix}7 & 7\\0 & 7\end{bmatrix}$

32. A basis of the plane is $\vec{v}_1 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$

Now apply the Gram-Schmidt process.

$$\vec{w}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$\vec{w}_2 = \frac{\vec{v}_2 - (\vec{w}_1\cdot\vec{v}_2)\vec{w}_1}{\text{length}} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

Your solution may be different if you start with a different basis $\vec{v}_1, \vec{v}_2$ of the plane.

39. $\vec{w}_1 = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{w}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}$, $\vec{w}_3 = \vec{w}_1 \times \vec{w}_2 = \frac{1}{\sqrt{42}}\begin{bmatrix}-5\\4\\-1\end{bmatrix}$

Section 5.3

6. Yes! By Facts 5.3.9b and 5.3.7, $(A^T)^{-1} = (A^{-1})^T = (A^T)^T$.
The equation $(A^T)^{-1} = (A^T)^T$ shows that A^T is orthogonal, again by Fact 5.3.7.

Yes! If A is orthogonal, then so is A^T, by Exercise 6. Since the columns of A are orthonormal, so are the rows of A^T.

8. a. No! As a counterexample, consider $A = \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$ (see Exercise 4).

b. Yes! More generally, if A and B are $n \times n$ matrices such that $BA = I_n$, then $AB = I_n$, by Fact 2.4.9c.

14. By Fact 5.3.9a, $(A^TA)^T = A^T(A^T)^T = A^TA$.
The matrix AA^T is symmetric as well: $(AA^T)^T = (A^T)^TA^T = AA^T$.

18. a. The general form of a skew-symmetric 3×3 matrix is $A = \begin{bmatrix}0 & a & b\\-a & 0 & c\\-b & -c & 0\end{bmatrix}$ with

$A^2 = \begin{bmatrix}-a^2-b^2 & -bc & ac\\-bc & -a^2-c^2 & -ab\\ac & -ab & -b^2-c^2\end{bmatrix}$ a symmetric matrix.

b. By Fact 5.3.9.a, $(A^2)^T = (A^T)^2 = (-A)^2 = A^2$, so that A^2 is symmetric.

34. To write the general form of a skew-symmetric $n \times n$ matrix A, we can place arbitrary constants above the diagonal, the opposite entries below the diagonal ($a_{ij} = -a_{ji}$), and zeros on the diagonal (since $a_{ii} = -a_{ii}$). See Exercise 33 for the case $n = 3$. Thus the dimension of the space equals the number of entries above the diagonal of an $n \times n$ matrix. In Exercise 35 we will see that there are $(n^2 - n)/2$ such entries. Thus $\dim(V) = (n^2 - n)/2$.

39. The kernel consists of all matrixes A such that $L(A) = A - A^T = 0$, that is, $A^T = A$; those are the symmetric matrices.
The value $L(A) = A - A^T$ is always skew-symmetric, since $(A - A^T)^T = A^T - (A^T)^T = -(A - A^T)$, and every skew-symmetric matrix B can be written as $B = L(\frac{1}{2}B)$. Thus the image consists of all skew-symmetric matrices.

16. If A is an $m \times n$ matrix, then
$\dim(\text{im}A)^{\perp} = m - \dim(\text{im}A) = m - \text{rank}(A)$
↑ Fact 5.4.2a ↑ Fact 3.3.8

and $\dim(\ker(A^T)) = m - \text{rank}(A^T)$
↑ Fact 3.3.5

It follows that $\text{rank}(A) = \text{rank}(A^T)$, as claimed.

17. Yes! By Fact 5.4.3, $\ker(A) = \ker(A^TA)$. Taking dimensions of both sides and using Fact 3.3.5, we find that $n - \text{rank}(A) = n - \text{rank}(A^TA)$; the claim follows.

18. Yes! By Exercise 17, $\text{rank}(A) = \text{rank}(A^TA)$. Substituting A^T for A in Exercise 17 and using Fact 5.3.9c, we find that $\text{rank}(A) = \text{rank}(A^T) = \text{rank}(AA^T)$. The claim follows.

19. $\vec{x}^* = (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, by Fact 5.4.7.

20. Using Fact 5.4.7, we find $\vec{x}^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\vec{b} - A\vec{x}^* = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Note that $\vec{b} - A\vec{x}^*$ is perpendicular to the two columns of A.

21. Using Fact 5.4.7, we find $\vec{x}^* = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\vec{b} - A\vec{x}^* = \begin{bmatrix} -12 \\ 36 \\ -18 \end{bmatrix}$, so that $\|\vec{b} - A\vec{x}^*\| = 42$.

22. Using Fact 5.4.7, we find $\vec{x}^* = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\vec{b} - A\vec{x}^* = \vec{0}$. This system is in fact consistent and $\vec{x}^*$ is the exact solution; the error $\|\vec{b} - A\vec{x}^*\|$ is 0.

23. Using Fact 5.4.7, we find $\vec{x}^* = \vec{0}$; here $\vec{b}$ is perpendicular to $\text{im}(A)$.

24. Using Fact 5.4.7, we find $\vec{x}^* = [2]$.

25. In this case, the normal equation $A^TA\vec{x} = A^T\vec{b}$ is $\begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$, which simplifies to $x_1 + 3x_2 = 1$, or $x_1 = 1 - 3x_2$. The solutions are of the form $\vec{x}^* = \begin{bmatrix} 1 - 3t \\ t \end{bmatrix}$, where t is an arbitrary constant.