True or False

1. T, by Fact 5.3.4.a

2. F. We have $(AB)^T = B^T A^T$, by Fact 5.3.9.a

3. T, since $(A+B)^T = A^T + B^T = A + B$

4. T, by Fact 5.3.4

5. F. Consider $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

6. T. First note that $A^T = A^{-1}$, by Fact 2.4.9. Thus A is orthogonal, by Fact 5.3.7

7. F. The correct formula is $\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{v})\vec{v}$, by Fact 2.2.5.

8. T, since $(7A)^T = 7A^T = 7A$.

9. F. Consider $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}$.

10. T, by Fact 5.3.9.b

11. T, since $(ABBA)^T = A^T B^T B^T A^T = ABBA$, by Fact 5.3.9.a

12. T, since $A^T B^T = (BA)^T = (AB)^T = B^T A^T$, by Fact 5.3.9.a

13. F. $\dim(V) + \dim(V^\perp) = 5$, by Fact 5.4.2.a. Thus one of the dimensions is even and the other odd.

14. T. Consider the QR factorization (Fact 5.2.2)

15. F. The Pythagorean Theorem holds for orthogonal vectors $\vec{x}, \vec{y}$ only (Fact 5.1.8)

16. T, by Definition 5.3.1. Note that $1 = \left\| A\left(\frac{1}{\|\vec{x}\|}\vec{x}\right)\right\| = \left\| \frac{1}{\|\vec{x}\|} A\vec{x} \right\| = \frac{1}{\|\vec{x}\|}\|A\vec{x}\|$ for all nonzero $\vec{x}$, so that $\|A\vec{x}\| = \|\vec{x}\|$.

17. T. If A is orthogonal, then $A^T A = I_n$, by Fact 5.3.7, so that $(A^T)^T = A = (A^T)^{-1}$. Again by Fact 5.3.7, this implies that matrix A^T is orthogonal.

18. F. Consider $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

19. T, since $\vec{x} \cdot (\text{proj}_V \vec{x}) = (\text{proj}_V \vec{x} + (\vec{x} - \text{proj}_V \vec{x})) \cdot \text{proj}_V \vec{x} = \|\text{proj}_V \vec{x}\|^2 \geq 0$. Note that $\vec{x} - \text{proj}_V \vec{x}$ is orthogonal to $\text{proj}_V \vec{x}$, by the definition of a projection.

20. F. It is required that the columns of A be orthonormal (Fact 5.3.10). As a counterexample, consider $A = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, with $AA^T = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$.

21. T, since the columns are unit vectors.

22. T. Use the Gram-Schmidt process to construct such a basis (Algorithm 5.2.1)

23. F. The columns fail to be unit vectors (use Fact 5.3.3.b)

24. T, by definition of an orthogonal projection (Fact 5.1.6).

25. F. As a counterexample, consider $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

26. T, by Fact 5.4.1.

27. T, by Fact 5.4.3.a.

28. F. Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, or any other symmetric matrix that fails to be orthogonal.

29. T. If $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is a symmetric matrix, then $A - xI_2 = \begin{bmatrix} a-x & b \\ b & c-x \end{bmatrix}$. This matrix fails to be invertible if (and only if) $\det(A - xI_2) = (a-x)(c-x) - b^2 = 0$. We use the quadratic formula to find the (real) solutions $x = \dfrac{a+c \pm \sqrt{(a+c)^2 - 4ac + 4b^2}}{2} = \dfrac{a+c \pm \sqrt{(a-c)^2 + 4b^2}}{2}$. Note that the discriminant $(a-c)^2 + 4b^2$ is positive or zero.

30. F. If A fails to be square, then $\frac{1}{2}(A - A^T)$ is undefined. The claim does hold for square matrices, however, since $\left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A) = -\left[\frac{1}{2}(A - A^T)\right]$.

31. F. Consider $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$.

32. T. By Fact 5.1.11, quantity $\cos(\alpha) = \dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}$ is positive, so that α is an acute angle.

33. T. In Fact 5.4.1, let $A = B^T$ to see that $(\operatorname{im}(B^T))^\perp = \ker(B)$. Now take the orthogonal complements of both sides and use Fact 5.4.2.b.

34. T, since $(A^TA)^T = A^T(A^T)^T = A^TA$, by Fact 5.3.9.a.

35. F. Verify that matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ are similar.

36. F. Consider $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The correct formula $\operatorname{im}(B) = \operatorname{im}(BB^T)$ follows from Facts 5.4.1 and 5.4.3.

37. T. We know that $A^T = A$ and $S^{-1} = S^T$. Now $(S^{-1}AS)^T = S^TA^T(S^{-1})^T = S^{-1}AS$, by Fact 5.3.9.a.

38. T. By Fact 5.4.3, we have $\ker(A) = \ker(A^T A)$. Replacing A by A^T in this formula, we find that $\ker(A^T) = \ker(AA^T)$. Now $\ker(A) = \ker(A^T A) = \ker(AA^T) = \ker(A^T)$.

39. T. Try $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, so that $A + B = \begin{bmatrix} 1+\cos\alpha & -\sin\alpha \\ \sin\alpha & 1+\cos\alpha \end{bmatrix}$. It is required that $\begin{bmatrix} 1+\cos\alpha \\ \sin a \end{bmatrix}$ and $\begin{bmatrix} -\sin\alpha \\ 1+\cos\alpha \end{bmatrix}$ be unit vectors, meaning that $1 + 2\cos\alpha + \cos^2\alpha + \sin^2\alpha = 2 + 2\cos\alpha = 1$, or $\cos\alpha = -\frac{1}{2}$, and $\sin\alpha = \pm\frac{\sqrt{3}}{2}$. Thus $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ is a solution.

40. F. Consider $A = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$, for example, representing a rotation-dilation.

41. T. We attempt to write $A = S + Q$, where S is symmetric and Q is skew-symmetric. Then $A^T = S^T + Q^T = S - Q$. Adding the equations $A = S + Q$ and $A^T = S - Q$ together gives $2S = A + A^T$ and $S = \frac{1}{2}(A + A^T)$. Similarly we find $Q = \frac{1}{2}(A - A^T)$. Check that the decomposition $A = S + Q = (\frac{1}{2}(A + A^T)) + (\frac{1}{2}(A - A^T))$ does the job.

42. T. Apply the Cauchy-Schwarz inequality (squared), $(\vec{x} \cdot \vec{y})^2 \leq \|\vec{x}\|^2 \|\vec{y}\|^2$, to $\vec{x} = \begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ \ldots \\ 1 \end{bmatrix}$ (all n entries are 1).

43. T. Let $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. We know that $AA^T = A^2$, or $\begin{bmatrix} x^2+y^2 & xz+yt \\ xz+yt & z^2+t^2 \end{bmatrix} = \begin{bmatrix} x^2+yz & xy+yt \\ zx+tz & yz+t^2 \end{bmatrix}$. We need to show that $y = z$. If $y \neq 0$, this follows from the equation $x^2 + y^2 = x^2 + yz$; if $z \neq 0$, it follows from $z^2 + t^2 = yz + t^2$; if both y and z are zero, we are all set.