35. $\det(A) = 1$ since there is only one pattern which does not contain a 0, and since the number of inversions in that pattern is $1 + 2 + 3 + \cdots + 99 = \dfrac{99 \cdot 100}{2} = 99 \cdot 50$, which is even.

37. Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$.

Then $\det(M) = a_1a_4c_1c_4 - a_2a_3c_1c_4 - a_1a_4c_2c_3 + a_2a_3c_2c_3 = \det(A)\det(C)$.

38. Not necessarily true; as a counter-example consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ so $\det(A) = \det(B) = \det(C) = \det(D) = 0$ hence $\det(A)\det(D) - \det(B)\det(C) = 0$ but $\det(M) = -1$.

39. Pick any pattern other than the diagonal pattern. Let a_{ii} be the left-most diagonal entry of the matrix such that a_{ii} is *not* in the chosen pattern. Then the chosen pattern must have an entry in row i to the right of a_{ii} and in column i below a_{ii}. This is the case since the rows above a_{ii} and the columns to the left of a_{ii} were already represented in the pattern by diagonal entries.

40. $\det(A) = 2(\cos^2\alpha + \sin^2\alpha) = 2$ so A is invertible for all values of α.

45. $\det(A^T) = \det\begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} = a_1a_4 - a_2a_3 = \det(A)$ so $\det(A^T) = \det(A) = k$.

52. The pattern containing all the 1000's has 4 inversions so it contributes $(1000)^{[illegible]} = 10^{[illegible]}$ to the value of the determinant. There are $5! - 1 = 119$ other patterns; the product associated with each of these patterns is less than $1000^3 \cdot 9^2 < 10^{11}$. Therefore we can say that $\det(A) > 0$.

6.2

6. There are many ways to do this problem; here is one possible approach:
Subtracting the second to last row from the last, we can make the last row into $[0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1]$.
Now expanding along the last row we see that $\det(M_n) = \det(M_{n-1})$.
Since $\det(M_1) = 1$ we can conclude that $\det(M_n) = 1$ for all n.

7. $\det(A) = 1$

8. Since $\vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, $T(\vec{x}) = 0$ only if $\vec{x}$ is a linear combination of the $\vec{v}_i$'s, (otherwise the matrix $[\vec{x}\,\vec{v}_2 \cdots \vec{v}_n]$ is invertible, and $T(\vec{x}) \neq 0$). Hence, the kernel of T is the span of $\vec{v}_2, \ldots, \vec{v}_n$, an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. The image of T is the real line $\mathbb{R}$ (since it must be 1-dimensional).

15. If a square matrix A has two equal columns, then its columns are linearly dependent, hence A is not invertible, and $\det(A) = 0$.

$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ $(b-a)t^2$ so $f(t)$ is a quadratic function of t.

23. Applying Fact 6.2.7 to the equation $AA^{-1} = I_n$ we see that $\det(A)\det(A^{-1}) = 1$.
The only way the product of the two integers $\det(A)$ and $\det(A^{-1})$ can be 1 is that they are both 1 or both -1. Therefore, $\det(A) = 1$ or $\det(A) = -1$.

24. $\det(A^TA) \underset{\substack{\uparrow \\ \text{Fact 6.2.7}}}{=} \det(A^T)\det(A) \underset{\substack{\uparrow \\ \text{Fact 6.2.1}}}{=} [\det(A)]^2 = 9$

25. $\det(A^TA) = \det(A^T)\det(A) = [\det(A)]^2 > 0$