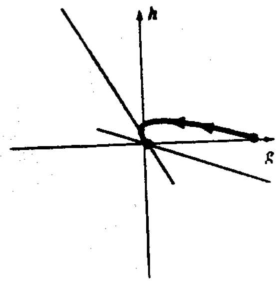
36. a. $\begin{bmatrix}0.978 & -0.006\\ 0.004 & 0.992\end{bmatrix}\begin{bmatrix}-1\\ 2\end{bmatrix} = \begin{bmatrix}-0.99\\ 1.98\end{bmatrix} = 0.99\begin{bmatrix}-1\\ 2\end{bmatrix}$; and $\begin{bmatrix}0.978 & -0.006\\ 0.004 & 0.992\end{bmatrix}\begin{bmatrix}3\\ -1\end{bmatrix} = \begin{bmatrix}2.94\\ -0.98\end{bmatrix} =$ $0.98\begin{bmatrix}3\\ -1\end{bmatrix}$. The eigenvalues are $\lambda_1 = 0.99$ and $\lambda_2 = 0.98$.

b. $\vec{x}_0 = \begin{bmatrix}g_0\\ l_0\end{bmatrix} = \begin{bmatrix}100\\ 0\end{bmatrix} = 20\begin{bmatrix}-1\\ 2\end{bmatrix} + 40\begin{bmatrix}3\\ -1\end{bmatrix}$ so $\vec{x}(t) = 20(0.99)^t\begin{bmatrix}-1\\ 2\end{bmatrix} + 40(0.98)^t\begin{bmatrix}3\\ -1\end{bmatrix}$, hence $g(t) = -20(0.99)^t + 120(0.98)^t$ and $h(t) = 40(0.99)^t - 40(0.98)^t$.



$h(t)$ first rises, then falls back to zero. $g(t)$ falls a little below zero, then goes back up to zero.

c. We set $g(t) = -20(0.99)^t + 120(0.98)^t = 0$.
Solving for t we get that $g(t) = 0$ for $t \approx 176$ minutes. (After $t = 176$, $g(t) < 0$).

37. a. $A = 5\begin{bmatrix}0.6 & 0.8\\ 0.8 & -0.6\end{bmatrix}$ is a scalar multiple of an orthogonal matrix. By Fact 7.1.2, the possible eigenvalues of the orthogonal matrix are ± 1, so that the possible eigenvalues of A are ± 5. In part b we see that both are indeed eigenvalues.

b. Solve $A\vec{v} = \pm 5\vec{v}$ to get $\vec{v}_1 = \begin{bmatrix}2\\ 1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}-1\\ 2\end{bmatrix}$.

38. a. We are given that
$n(t+1) = 2a(t)$
$a(t+1) = n(t) + a(t)$,
so that the matrix is $A = \begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}$.

b. $A\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}2\\ 2\end{bmatrix} = 2\begin{bmatrix}1\\ 1\end{bmatrix}$ and $A\begin{bmatrix}2\\ -1\end{bmatrix} = \begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix} = \begin{bmatrix}-2\\ 1\end{bmatrix} = -\begin{bmatrix}2\\ -1\end{bmatrix}$, hence 2 and -1 are the eigenvalues associated with $\begin{bmatrix}1\\ 1\end{bmatrix}$ and $\begin{bmatrix}2\\ -1\end{bmatrix}$ respectively.

c. We are given $\vec{x}_0 = \begin{bmatrix}n_0\\ a_0\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}$ so $\vec{x}_0 = \frac{1}{3}\begin{bmatrix}1\\ 1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}2\\ -1\end{bmatrix}$, and $\vec{x}(t) = \frac{1}{3}2^t\begin{bmatrix}1\\ 1\end{bmatrix} + \frac{1}{3}(-1)^t\begin{bmatrix}2\\ -1\end{bmatrix}$ (by Fact 7.1.3), hence $n(t) = \frac{1}{3}2^t + \frac{2}{3}(-1)^t$ and $a(t) = \frac{1}{3}2^t - \frac{1}{3}(-1)^t$.

39. Let λ be an eigenvalue of $S^{-1}AS$. Then for some nonzero vector $\vec{v}$, $S^{-1}AS\vec{v} = \lambda\vec{v}$, i.e., $AS\vec{v} = S\lambda\vec{v} = \lambda S\vec{v}$ so λ is an eigenvalue of A with eigenvector $S\vec{v}$.
Conversely, if α is an eigenvalue of A with eigenvector $\vec{w}$, then $A\vec{w} = \alpha\vec{w}$.
Therefore, $S^{-1}AS(S^{-1}\vec{w}) = S^{-1}A\vec{w} = S^{-1}\alpha\vec{w} = \alpha S^{-1}\vec{w}$, so $S^{-1}\vec{w}$ is an eigenvector of $S^{-1}AS$ with eigenvalue α.

40. $(A^2 + 2A + 3I_n)\vec{v} = A^2\vec{v} + 2A\vec{v} + 3I_n\vec{v} = 4^2\vec{v} + 2\cdot 4\vec{v} + 3\vec{v} = (16 + 8 + 3)\vec{v} = 27\vec{v}$ so $\vec{v}$ is an eigenvector of $A^2 + 2A + 3I_n$ with eigenvalue 27.

7.2

4. $\det(\lambda I_2 - A) = \det\begin{bmatrix}\lambda & -4\\ 1 & \lambda - 4\end{bmatrix} = \lambda(\lambda - 4) + 4 = (\lambda - 2)^2 = 0$ so $\lambda = 2$ with algebraic multiplicity 2.

5. $\det(\lambda I_2 - A) = \det\begin{bmatrix}\lambda - 11 & 15\\ -6 & \lambda + 7\end{bmatrix} = \lambda^2 - 4\lambda + 13$ so $\det(\lambda I_2 - A) = 0$ for no real λ.

6. $\det(\lambda I_2 - A) = \det\begin{bmatrix}\lambda - 1 & -2\\ -3 & \lambda - 4\end{bmatrix} = \lambda^2 - 5\lambda - 2 = 0$ so $\lambda_{1,2} = \dfrac{5 \pm \sqrt{33}}{2}$.

7. $\lambda = 1$ with algebraic multiplicity 3, by Fact 7.2.2.

8. $f_A(\lambda) = \lambda^2(\lambda + 3)$ so
$\lambda_1 = 0$ (Algebraic multiplicity 2)
$\lambda_2 = -3$.

14. $f_A(\lambda) = \det(\lambda I_2 - B)\det(\lambda I_2 - D)$ (see Exercise 6.1.37).
The eigenvalues of A are the eigenvalues of B and D. The eigenvalues of C are irrelevant.

15. $f_A(\lambda) = \lambda^2 - 2\lambda + (1 - k) = 0$ if $\lambda_{1,2} = \dfrac{2 \pm \sqrt{4 - 4(1 - k)}}{2} = 1 \pm \sqrt{k}$
The matrix has 2 distinct real eigenvalues when $k > 0$, no real eigenvalues when $k < 0$.

16. $f_A(\lambda) = \lambda^2 - (a + c)\lambda + (ac - b^2)$

28. a. $w(t + 1) = 0.8w(t) + 0.1m(t)$
$m(t + 1) = 0.2w(t) + 0.9m(t)$
so $A = \begin{bmatrix}0.8 & 0.1\\ 0.2 & 0.9\end{bmatrix}$ which is a regular transition matrix since its columns sum to 1 and its entries are positive.

b. The eigenvectors of A are $\begin{bmatrix}0.1\\ 0.2\end{bmatrix}$ or $\begin{bmatrix}1\\ 2\end{bmatrix}$ with $\lambda_1 = 1$, and $\begin{bmatrix}1\\ -1\end{bmatrix}$ with $\lambda_2 = 0.7$.

$\vec{x}_0 = \begin{bmatrix}1200\\ 0\end{bmatrix} = 400\begin{bmatrix}1\\ 2\end{bmatrix} + 800\begin{bmatrix}1\\ -1\end{bmatrix}$ so $\vec{x}(t) = 400\begin{bmatrix}1\\ 2\end{bmatrix} + 800(0.7)^t\begin{bmatrix}1\\ -1\end{bmatrix}$ or
$w(t) = 400 + 800(0.7)^t$
$m(t) = 800 - 800(0.7)^t$.

c. As $t \to \infty$, $w(t) \to 400$ so Wipfs won't have to close the store.

12. $\lambda_1 = \lambda_2 = \lambda_3 = 1$. $E_1 = \text{span}\begin{bmatrix}1\\0\\0\end{bmatrix}$, no eigenbasis

20. For $\lambda_1 = 1$, $E_1 = \ker\begin{bmatrix}0&a&b\\0&0&c\\0&0&1\end{bmatrix} = \ker\begin{bmatrix}0&a&0\\0&0&1\\0&0&0\end{bmatrix}$ so if $a = 0$ then E_1 is 2-dimensional, otherwise it is 1-dimensional.

For $\lambda_2 = 2$, $E_2 = \ker\begin{bmatrix}1&-a&-b\\0&1&-c\\0&0&0\end{bmatrix}$ so E_2 is 1-dimensional.

Hence, there is an eigenbasis if $a = 0$.

22. We want A such that $A\vec{e}_1 = 7\vec{e}_1$ and $A\vec{e}_2 = 7\vec{e}_2$ hence $A = \begin{bmatrix}7&0\\0&7\end{bmatrix}$.

25. If λ is an eigenvalue of A, then $E_\lambda = \ker(\lambda I_3 - A) = \ker\begin{bmatrix}\lambda&-1&0\\0&\lambda&-1\\-a&-b&\lambda-c\end{bmatrix}$.

The row-reduced echelon form of the above matrix will contain two leading 1's hence E_λ is always 1-dimensional, i.e. the geometric multiplicity of λ is 1.

16. Diagonalizable. The eigenvalues are 3,2,1, with associated eigenvectors $\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$. If we let $S = \begin{bmatrix}2&1&0\\0&0&1\\1&1&0\end{bmatrix}$, then $S^{-1}AS = D = \begin{bmatrix}3&0&0\\0&2&0\\0&0&1\end{bmatrix}$.

35. Matrix $\begin{bmatrix}-1&6\\-2&6\end{bmatrix}$ has the eigenvalues 3 and 2. If $\vec{v}$ and $\vec{w}$ are associated eigenvectors, and if we let $S = [\vec{v}\,\vec{w}]$, then $S^{-1}\begin{bmatrix}-1&6\\-2&6\end{bmatrix}S = \begin{bmatrix}3&0\\0&2\end{bmatrix}$, so that matrix $\begin{bmatrix}-1&6\\-2&6\end{bmatrix}$ is indeed similar to $\begin{bmatrix}3&0\\0&2\end{bmatrix}$.

36. Yes. The matrices $\begin{bmatrix}-1&6\\-2&6\end{bmatrix}$ and $\begin{bmatrix}1&2\\-1&4\end{bmatrix}$ both have the eigenvalues 3 and 2, so that each of them is similar to the diagonal matrix $\begin{bmatrix}3&0\\0&2\end{bmatrix}$, by Algorithm 7.4.4. Thus $\begin{bmatrix}-1&6\\-2&6\end{bmatrix}$ is similar to $\begin{bmatrix}1&2\\-1&4\end{bmatrix}$, by parts b and c of Fact 3.4.6.

37. Yes. Matrices A and B have the same characteristic polynomial, $\lambda^2 - 7\lambda + 7$, so that they have the same two distinct real eigenvalues $\lambda_{1,2} = \frac{7 \pm \sqrt{21}}{2}$. Thus both A and B are similar to the diagonal matrix $\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}$, by Algorithm 7.4.4. Therefore A is similar to B, by parts b and c of Fact 3.4.6.

55. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, for example.

15. $\lambda_{1,2} = 1 \pm \frac{1}{10}\sqrt{k}$

If $k \geq 0$ then $\lambda_1 = 1 + \frac{1}{10}\sqrt{k} \geq 1$. If $k < 0$ then $|\lambda_1| = |\lambda_2| > 1$. Thus, the zero state isn't a stable equilibrium for any real k.

28. Not stable, since if λ is an eigenvalue of A, then $(\lambda - 2)$ is an eigenvalue of $(A - 2I_n)$ and $|\lambda - 2| > 1$.

30. Consider dynamical systems $\vec{x}(t+1) = A^2\vec{x}(t)$ and $\vec{y}(t+1) = A\vec{y}(t)$ with equal initial values, $\vec{x}(0) = \vec{y}(0)$. Then $\vec{x}(t) = \vec{y}(2t)$ for all positive integers t. We know that $\lim_{t\to\infty} \vec{y}(t) = \vec{0}$; thus $\lim_{t\to\infty} \vec{x}(t) = \vec{0}$, proving that the zero state is a stable equilibrium of the system $\vec{x}(t+1) = A^2\vec{x}(t)$.

40. Note that A can be partitioned as $A = \begin{bmatrix} B & -C^T \\ C & B^T \end{bmatrix}$, where B and C are rotation-dilation matrices. Also note that $BC = CB, B^TB = (p^2+q^2)I_2$, and $C^TC = (r^2+s^2)I_2$.

a. $A^TA = \begin{bmatrix} B^T & C^T \\ -C & B \end{bmatrix}\begin{bmatrix} B & -C^T \\ C & B^T \end{bmatrix} = (p^2+q^2+r^2+s^2)I_4$

b. By part a, $A^{-1} = \frac{1}{p^2+q^2+r^2+s^2}A^T$ if $A \neq 0$.

c. $(\det A)^2 = (p^2+q^2+r^2+s^2)^4$, by part a, so that $\det A = \pm(p^2+q^2+r^2+s^2)^2$.
The diagonal pattern makes the contribution $+p^4$, so that $\det A = (p^2+q^2+r^2+s^2)^2$.

d. Consider $\det(\lambda I_4 - A)$. Note that the matrix $\lambda I_4 - A$ has the same "format" as A, with p replaced by $\lambda - p$ and q, r, s by $-q, -r, -s$, respectively. By part c, $\det(\lambda I_4 - A) = ((\lambda-p)^2+q^2+r^2+s^2)^2 = 0$ when

$(\lambda-p)^2 = -q^2-r^2-s^2$

$\lambda - p = \pm i\sqrt{q^2+r^2+s^2}$

$\lambda_{1,2} = p \pm i\sqrt{q^2+r^2+s^2}$

Each of these eigenvalues has algebraic multiplicity 2 (if $q = r = s = 0$ then $\lambda = p$ has algebraic multiplicity 4).

e. By part a we can write $A = \sqrt{p^2+q^2+r^2+s^2}\underbrace{\left(\frac{1}{\sqrt{p^2+q^2+r^2+s^2}}A\right)}_{S}$, where S is orthogonal.

Therefore, $\|A\vec{x}\| = \|\sqrt{p^2+q^2+r^2+s^2}(S\vec{x})\| = \sqrt{p^2+q^2+r^2+s^2}\|\vec{x}\|$.

f. Let $A = \begin{bmatrix} 3 & -3 & -4 & -5 \\ 3 & 3 & 5 & -4 \\ 4 & -5 & 3 & 3 \\ 5 & 4 & -3 & 3 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}$; then $A\vec{x} = \begin{bmatrix} -39 \\ 13 \\ 18 \\ 13 \end{bmatrix}$.

By part e, $\|A\vec{x}\|^2 = (3^2+3^2+4^2+5^2)\|\vec{x}\|^2$, or
$39^2+13^2+18^2+13^2 = (3^2+3^2+4^2+5^2)(1^2+2^2+4^2+4^2)$, as desired.

g. Any positive integer m can be written as $m = p_1p_2\dots p_n$. Using part f repeatedly we see that the numbers $p_1, p_1p_2, p_1p_2p_3, \dots, p_1p_2p_3\cdots p_{n-1}$, and finally $m = p_1\cdots p_n$ can be expressed as the sums of four squares.