1. For each of the following, circle T for true or F for false. No explanation is necessary.

(a)   T   F   Every diagonalizable matrix is symmetric.

(b)   T   F   Every real 3×3 matrix having $1 + i$ as an eigenvalue is diagonalizable over the complex numbers.

(c)   T   F   If A is an invertible real 3×3 matrix, $A^T A$ is positive definite.

(d)   T   F   If S is the set of real 3×3 matrices whose image is the xy-plane then S is a subspace of $\mathbf{R}^{3 \times 3}$.

(e)   T   F   If A is a nonzero diagonalizable 4×4 matrix, then A^4 is nonzero.

(f)   T   F   There exists a real 2×2 matrix A such that $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

(g)   T   F   There exist invertible 2×2 matrices A and B such that $\det(A + B) = \det(A) + \det(B)$.

(h)   T   F   If there is a real eigenbasis of $\mathbf{R}^n$ for a real $n \times n$ matrix A, there is also a real eigenbasis of $\mathbf{R}^n$ for A^T.

(i)   T   F   Every 2×2 matrix which is not invertible is diagonalizable.

(j)   T   F   If the entries of a 2×2 real matrix are all positive but less than 0.9, then $A^t \vec{v}$ tends to $\vec{0}$ as t tends to ∞ for all $\vec{v} \in \mathbf{R}^2$.

2. Find a single 3×3 matrix A such that ALL of the following statements are true. (The statements must all be true for ONE choice of the matrix A.)

- The kernel of A is the span of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$;

- $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for A;

- $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is in the image of A.

3. Suppose $T : \mathbf{R}^3 \to \mathbf{R}^5$ and $S : \mathbf{R}^5 \to \mathbf{R}^3$ are linear transformations, each of which has rank 3.

Answer each of the following questions, in each case both JUSTIFYING your answer and GIVING AN EXAMPLE of transformations S and T realizing your stated maximum/minimum.

(a) What is the maximum possible value of the dimension of the kernel of TS? [Remember to give an example!]

(b) What is the minimum possible value of the dimension of the image of ST? [Remember to give an example!]

4. Let A be a symmetric $n \times n$ matrix with real entries and with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be a corresponding orthonormal eigenbasis. Let

$$\vec{x} = b_1 \vec{v}_1 + \ldots + b_n \vec{v}_n$$

be some vector in $\mathbf{R}^n$.

   (a) Express $\vec{x} \cdot \vec{x}$ in terms of the b_i's.
   (b) Express $\vec{x}^T A \vec{x}$ in terms of the b_i's and λ_i's.
   (c) Suppose that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. What is the maximum value of

$$\frac{\vec{x}^T A \vec{x}}{\vec{x} \cdot \vec{x}}$$

   as $\vec{x}$ varies over all nonzero vectors in $\mathbf{R}^n$?

5. Find the volume of the tetrahedron (pyramid with triangular base) whose four vertices are the points

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}.$$

in $\mathbf{R}^3$.

[Hint: the volume of a tetrahedron is one-third the area of the base multiplied by the height. For example, the tetrahedron whose vertices are $\vec{0}$ and the standard basis elements $\vec{e}_1, \vec{e}_2, \vec{e}_3$ has volume $1/6$.]

6. Find the matrix (with respect to the standard basis) for the following linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^3$: rotation by $90°$ about the line spanned by $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, in a clockwise direction as viewed when facing the origin from the point $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

7. (a) Find the eigenvalues of the matrix $A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}$.

(b) $\vec{0}$ is an equilibrium point for the linear system

$$\frac{d\vec{x}}{dt} = A\vec{x}.$$

Is it stable? Why?

(c) $\vec{0}$ is an equilibrium point for the linear system

$$\frac{d\vec{x}}{dt} = B\vec{x},$$

where

$$B = \begin{bmatrix} 0 & -1 & -9 & -9 & -8 \\ 0 & 0 & 0 & -1 & -9 \\ 5 & 0 & 5 & 0 & -5 \\ 1 & 9 & 0 & 0 & 0 \\ 1 & 9 & 9 & 8 & 0 \end{bmatrix}.$$

Is it stable? Why?

[Hint: You are strongly advised not to try to find the eigenvalues of B, nor to make any other lengthy computation.]

8. Match the five systems of differential equations with the five descriptions. (Next to each description write the letter denoting the system which it describes.) Each description applies to only one equation.

(a) $\frac{dx}{dt} = 3x - 5y, \frac{dy}{dt} = 2x - 3y$

(b) $\frac{dx}{dt} = -4y + 2x^2 + 2x^3, \frac{dy}{dt} = 4y(1 - x^2)$

(c) $\frac{dx}{dt} = -x + 2y - y^2, \frac{dy}{dt} = 3x - y - xy - y^2$

(d) $\frac{dx}{dt} = 3x - 5y, \frac{dy}{dt} = x^2 + y^2 + 2$

(e) $\frac{dx}{dt} = 2y(x - y) - x, \frac{dy}{dt} = y(x - y) - y$

_____(i) The equation has a stable equilibrium at $x = 1, y = 1$.

_____(ii) The equation has an unstable equilibrium at $x = 1, y = 1$.

_____(iii) The equation has a non-constant solution which stays on the line $x = y$.

_____(iv) The equation has a closed periodic orbit.

_____(v) The equation has no equilibria.

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9. Let V be the linear space of antisymmetric 3×3 real matrices; that is, V is the subspace of $\mathbf{R}^{3\times 3}$ consisting of those matrices A such that $A^T = -A$.

Let S be the function on V defined by
$$S(A) = BAB^T$$
where B is the fixed matrix
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(a) What is the dimension of V?

(b) Show that S is a linear transformation from V to itself.

(c) Find a basis for the kernel of S and a basis for the image of S. Be sure to say which is which.

(d) Find an eigenbasis of V for S.

(e) What are the eigenvalues of S?