Last Name:______________________________

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Mathematics 21b

Final Exam Solutions
May 17, 2002

Your Section (circle one):

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MWF 10	MWF 11	MWF 12	TuTh 10	TuTh 11:30

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	12	
6	11	
7	9	
8	18	
Total	100	

The exam will last 3 hours.
No calculators are allowed.
Justify your answers carefully (except in Questions 1).
Write your final answers in the spaces provided.

(1) True or False (no explanation is necessary).

T **(F)** **:** There are 4×4 matrices A and B of ranks 3 and 1 respectively such that AB has rank 2.

Note that AB has rank at most equal to the lesser of the ranks of A and B.

(T) **F** **:** There are 4×4 matrices A and B of ranks 3 and 1 respectively such that AB has rank 0.

Consider for instance $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(T) **F** **:** If A is an invertible 4×4 matrix then the unique least squares solution to $A\vec{x} = \vec{b}$ is $A^{-1}\vec{b}$.

In general, the unique least squares solution is $\vec{x} = (A^T A)^{-1} A^T \vec{b}$; in the case where A and hence A^T are invertible this simplifies to $\vec{x} = A^{-1}(A^T)^{-1}A^T\vec{b} = A^{-1}\vec{b}$.

T **(F)** **:** The matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

has determinant 0.

There are many ways to see this, one of which is that the reduced row-echelon form of the matrix does not has no rows of zeros (in fact the rref is I_3).

(T) **F** **:** The matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

has determinant 0.

If we try to find the reduced row-echelon form of this matrix, we get, after subtracting row 1 from row 2 and row 4 from row 3, the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

Subtracting row 2 from row 3 then gives a row of zeros, so the rref has a row of zeros and hence the determinant of the matrix is zero.

(T) F : If A is a 2×2 matrix representing a shear then A has only one eigenvalue.

By definition a shear preserves one line L and shifts all other vectors parallel to L; in fact the lone eigenvalue is 1.

(T) F : If A is a 4×4 matrix then A and A^T have the same eigenvalues.

For, $\det B = \det B^T$ for any matrix B, so in particular $\det A - \lambda I_4 = \det(A - \lambda I_4)^T = \det A^T - \lambda I_4^T = \det A^T - \lambda I_4$. Since the characteristic polynomials for the two matrices are the same, they have the same eigenvalues.

T (F) : If A is a 4×4 matrix then A and A^T have the same eigenvectors.

For instance, if $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ then $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of A but not of A^T, and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector of A^T but not of A (in this example all of the other eigenvectors of the matrices are the same).

T (F) : $\vec{0}$ is a stable equilibrium for the discrete dynamical system

$$\vec{x}(n+1) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x}(n).$$

The eigenvalues of the system are $\lambda = 1 \pm i$. Since both eigenvalues have modulus $2 > 1$, the system spirals out to infinity and $\vec{0}$ is not a stable equilibrium.

T (F) : The map $T : C^\infty \to C^\infty$ given by

$$T(f)(t) = t + f(t)$$

is linear.

T is not linear under addition: $T(f+g) = t + f(t) + g(t) \neq t + f(t) + t + g(t) = T(f) + T(g)$.

(2) Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & -1 & -1 \end{bmatrix}.$$

(a) Find the reduced row echelon form of A.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{+\text{V}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{-\text{IV}}\text{rearrange} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & -2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\text{II}} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{+2\text{II}} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\text{III}} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find a basis of the kernel of A.

The basis of $\ker A$ is the same as the basis of $\ker(\operatorname{rref} A)$. From part a) we see that $\ker A$ is given by

$$\ker A = \begin{bmatrix} t-s \\ 0 \\ s \\ -t \\ t \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

so that a basis for $\ker A$ is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(c) Find a basis of the image of A.

Note that the image of A is spanned by the columns of A that have leading ones in $\operatorname{rref} A$, so

$$\operatorname{Im} A = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right).$$

Since these three vectors are linearly dependent, they in fact form a basis for the image of A.

(d) Find a 2×4 matrix B with $\ker B = \mathrm{Im} A$.

We would like $\ker B$ to be spanned by $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$. This is easiest to do if we allow B to already be in reduced row-echelon form, in which case we see that

$$B = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is one matrix with the desired kernel (Note: many matrices are possible for this question).

(3) Let

$$A = \begin{bmatrix} -8 & 4 & -1 \\ 4 & 1 & 5 \\ 1 & 1 & -1 \end{bmatrix}.$$

(a) Find the QR factorization of A.

First we orthogonalize the columns of A:

$$\vec{w}_1 = \frac{\vec{v}_1}{||\vec{v}_1||} = \frac{1}{9}\begin{bmatrix} -8 \\ 4 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \frac{\vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1}{||\vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1||} = \frac{1}{3}\left(\frac{1}{3}\begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix}\right) = \frac{1}{9}\begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix}$$

$$= \frac{1}{3}(\vec{v}_2 + \frac{1}{3}\vec{v}_1)$$

$$\vec{w}_3 = \frac{\vec{v}_3 - (\vec{v}_3 \cdot \vec{w}_1)\vec{w}_1 - (\vec{v}_3 \cdot \vec{w}_2)\vec{w}_2}{||\vec{v}_3 - (\vec{v}_3 \cdot \vec{w}_1)\vec{w}_1 - (\vec{v}_3 \cdot \vec{w}_2)\vec{w}_2||}$$

$$= \frac{1}{3}\left(\begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -8 \\ 4 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix}\right) = \frac{1}{9}\begin{bmatrix} 1 \\ 4 \\ -8 \end{bmatrix}$$

$$= \frac{1}{3}\left(-\frac{2}{3}\vec{v}_1 - \vec{v}_2 + \vec{v}_3\right)$$

From this we see that

$$\vec{v}_1 = 9\vec{w}_1$$

$$\vec{v}_2 = 3\vec{w}_2 - \frac{1}{3}\vec{v}_1 = -3\vec{w}_1 + 3\vec{w}_2$$

$$\vec{v}_3 = 3\vec{w}_3 + \frac{2}{3}\vec{v}_1 + \vec{v}_2 = 3\vec{w}_1 + 3\vec{w}_2 + 3\vec{w}_3.$$

The columns of Q are then the orthonormal vectors $\vec{w}_1, \vec{w}_2, \vec{w}_3$, and the columns of R are the coefficients of the linear combinations needed to write the $\vec{v}_i$ in terms of the $\vec{w}_j$, so that

$$Q = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 4 & 7 & 4 \\ 1 & 4 & -8 \end{bmatrix},$$

$$R = \begin{bmatrix} 9 & -3 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{bmatrix}, \text{ and}$$

$$A = QR$$

as desired.

(b) Solve the equation

$$A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}.$$

[HINT: You may use any method you wish, but it is probably quickest to use your answer from part (a).]

Since $A = QR$, where Q is orthogonal, we can use the fact that $Q^{-1} = Q^T$ to solve this equation relatively easily. Furthermore, since Q is symmetric we see that in fact $Q^{-1} = Q$, and we get

$$\begin{aligned} QR\vec{x} &= \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \\ \Rightarrow R\vec{x} &= Q^{-1}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 4 & 7 & 4 \\ 1 & 4 & -8 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 9 & -3 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} \Rightarrow \begin{array}{l} x_3 = -1 \\ x_2 = 2 \\ x_1 = 1. \end{array}$$

So the solution is

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

(4) (a) Find a matrix S and a diagonal matrix D such that

$$S^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} S = D.$$

We first find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$:

$$\begin{aligned} \det(A - \lambda I_3) &= (1-\lambda)\left((1-\lambda)^2 - 1\right) \\ &= (1-\lambda)(\lambda^2 - 2\lambda) \end{aligned}$$

$$\Rightarrow \lambda = \{1, 0, 2\}$$

Since the three eigenvalues are distinct we know that we may find an eigenbasis for A; the eigenvectors are by inspection

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{v}_0 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

So, letting S be the matrix whose columns are given by the eigenvectors we have that

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

where $D = S^{-1}AS$.

(b) The matrix

$$A = \begin{bmatrix} 7/9 & 4/9 & 0 \\ 4/9 & 0 & 4/9 \\ 0 & 4/9 & -7/9 \end{bmatrix}$$

has eigenvalues -1, 0 and 1. Find real numbers a, b and c such that

$$a(A - bI_3)(A - cI_3)$$

represents orthogonal projection onto the kernel of A. [HINT: Consider the effect on the eigenvectors of A. This question requires no calculation.]

Note that the kernel of A is spanned by the eigenvector $\vec{v}_0$. Thus the orthogonal projection onto ker$A$ should leave $\vec{v}_0$ unchanged while sending $\vec{v}_1$ to $(\vec{v}_1 \cdot \vec{v}_0)\vec{v}_0$ and $\vec{v}_{-1}$ to $(\vec{v}_{-1} \cdot \vec{v}_0)\vec{v}_0$ (assuming that $\vec{v}_0$ is normalized). Now, we could find the (normalized) eigenvectors with relatively little difficulty by solving the equation $A\vec{x} = \lambda\vec{x}$. This is, however, unnecessary, because A is symmetric, and so the eigenvectors corresponding to different eigenvalues are orthogonal. Therefore we need only send $\vec{v}_0 \to \vec{v}_0$, $\vec{v}_{\pm 1} \to 0$. This is accomplished for $b = 1$, $c = -1$, and $a = -1$.

(5) (a) Find a solution $f(t)$ of

$$f'(t) + 3f(t) = e^{-2t}$$

which also satisfies $f(0) = 0$.

First solve the corresponding homogeneous equation

$$f'(t) + 3f(t) = 0$$

which has solution $f_h(t) = Ce^{-3t}$. The particular solution we guess to be of the form $f_p(t) = De^{-2t}$; substituting this into the original equation gives

$$-2De^{-2t} + 3De^{-2t} = e^{-2t}$$
$$\Rightarrow D = 1$$

so that $f_p(t) = e^{-2t}$. The general solution is then

$$f_g(t) = f_p(t) + f_h(t) = e^{-2t} + Ce^{-3t}.$$

Imposing the initial condition $f(0) = 0$, we get that $C = -1$, so that

$$f(t) = e^{-2t} - e^{-3t}.$$

(b) Find a solution $f(t)$ of

$$f''(t) + 4f'(t) + 3f(t) = 1$$

which also satisfies $f(0) = 1/3$ and $f(1) = 1/3 + 1/e^3 - 1/e$.

The characteristic polynomial of the corresponding homogeneous equation is $\lambda^2 + 4\lambda + 3 = 0 \Rightarrow \lambda = -1, -3$, so that the homogeneous solution is $f_h(t) = Ae{-}3t + Be^{-t}$. For the particular solution note that $f_p(t) = 1/3$ will work (one could also substitute in a more generic polynomial, say $c_1x^2 + c_2x + c_3$, and solve for the c_i). The general solution is then

$$f_g(t) = \frac{1}{3} + Ae^{-3t} + Be^{-t}.$$

The condition $f(0) = 1/3$ implies $A = -B$, and from the second condition we obtain

$$f(1) = \frac{1}{3} + Ae^{-3t} - Ae^{-t} = \frac{1}{3} + \frac{1}{e^3} - \frac{1}{e}$$
$$\Rightarrow A = 1.$$

So the final solution is

$$f(t) = \frac{1}{3} + e^{-3t} - e^{-t}.$$

(6) (a) Find the Fourier series of the function $\sin(x/2) \in C[-\pi, \pi]$.
[We remind you of the following formulae:

$$\begin{aligned} 2\sin A \sin B &= \cos(A-B) - \cos(A+B) \\ 2\cos A \sin B &= \sin(A+B) - \sin(A-B) \\ 2\cos A \cos B &= \cos(A+B) + \cos(A-B).] \end{aligned}$$

Note that $\sin(x/2)$ is an odd function, so the coefficients of $1/\sqrt{2}$ and of all of the cosine terms in its Fourier expansion will be zero. The coefficients of the sine terms we find as follows:

$$\begin{aligned} c_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin\frac{x}{2} \sin nx\, dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos(n-\tfrac{1}{2})x - \cos(n+\tfrac{1}{2})x)dx \\ &= \frac{1}{2\pi}\left[\frac{1}{n-1/2}\sin(n-\tfrac{1}{2})x - \frac{1}{n+1/2}\sin(n+\tfrac{1}{2})x\right]_{-\pi}^{\pi} \\ &= \frac{(-1)^{n-1}}{\pi}\left[\frac{1}{n-1/2} + \frac{1}{n+1/2}\right] \\ &= \frac{1}{\pi}\frac{(-1)^{n-1}8n}{4n^2-1}. \end{aligned}$$

Thus

$$\sin\frac{x}{2} = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}8n}{4n^2-1} \sin nx.$$

(b) Solve the heat equation

$$\partial T/\partial t = \partial^2 T/\partial x^2$$

in the region $t \geq 0$ and $0 \leq x \leq \pi$ and subject to the boundary conditions $T(0,t) = 0 = T(\pi,t)$ and

$$T(x,0) = \sin(x/2) - x/\pi.$$

[HINT: You may assume that $x \in C[-\pi,\pi]$ has Fourier expansion

$$x = 2\sum_{n=1}^{\infty}((-1)^{n-1}/n)\sin(nx).]$$

We begin by supposing that the solution to the heat equation is separable, $T(x,t) = u(x)v(t)$. Substituting this form of T into the heat equation and separating variables gives

$$\frac{v'}{v} = \frac{u''}{u} = k^2$$

where k^2 is an arbitrary constant. From this we see that $v(t) = Ce^{k^2t}$, and that there are three possiblities for $u(x)$:

(i) $k^2 > 0 \Rightarrow u(x) = Ae^{kt} + Be^{-kt}$ (k is real)
(ii) $k^2 = 0 \Rightarrow u(x) = A + Bt$
(iii) $k^2 < 0 \Rightarrow u(x) = A\sin(|k|t) + B\cos(|k|t)$ (k is purely imaginary).

Examining the boundary conditions $T(0,t) = 0 = T(\pi,t)$, we see that only case (iii) gives a nontrivial solution, and furthermore that $B = 0$ and $k^2 = -n^2$, where $n \in \mathbb{Z}$. So our solution so far is

$$T(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2t}\sin nx.$$

We now need to fit the final boundary condition, $T(x,0) = \sin(x/2) - x/\pi$ and solve for the coefficients c_i. To do this we write $\sin(x/2) - x/\pi$ as a Fourier sine series and equate coefficients when $t = 0$:

$$\begin{aligned}
T(x,0) &= \sum_{n=1}^{\infty} c_n e^0 \sin nx \\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}8n}{4n^2-1}\sin nx - \frac{2}{\pi}\sum_{n=1}^{\infty}((-1)^{n-1}/n)\sin nx \\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{8n}{4n^2-1} - \frac{2}{n}\right)\sin nx \\
\Rightarrow c_n &= \frac{(-1)^{n-1}}{\pi}\left(\frac{8n}{4n^2-1} - \frac{2}{n}\right).
\end{aligned}$$

The solution is then

$$T(x,t) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{8n}{4n^2-1} - \frac{2}{n}\right)e^{-n^2t}\sin nx.$$

(7) (a) Find the Fourier series of the function $\sin|x/2| \in C[-\pi,\pi]$. Note the absolute value in this expression.
[We remind you of the following formulae:

$$\begin{aligned} 2\sin A\sin B &= \cos(A-B)-\cos(A+B) \\ 2\cos A\sin B &= \sin(A+B)-\sin(A-B) \\ 2\cos A\cos B &= \cos(A+B)+\cos(A-B).] \end{aligned}$$

Note that $\sin|x/2|$ is an *even* function, so that the sine terms in its Fourier expansion will be zero. We now compute the Fourier coefficients of the constant and cosine terms:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\sqrt{2}}\sin\left|\frac{x}{2}\right|\,dx \\ &= \frac{2}{\pi}\int_0^{\pi}\frac{1}{\sqrt{2}}\sin\frac{x}{2}\,dx \\ &= -\frac{2\sqrt{2}}{\pi}\left[\cos\frac{x}{2}\right]_0^{\pi} \\ &= \frac{2\sqrt{2}}{\pi} \\ d_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos nx\sin\left|\frac{x}{2}\right|\,dx \\ &= \frac{2}{\pi}\int_0^{\pi}\cos nx\sin\frac{x}{2},\,dx \\ &= \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}(\sin(n+\tfrac{1}{2})x-\sin(n-\tfrac{1}{2})x)dx \\ &= \frac{1}{\pi}\left[-\frac{1}{n+1/2}\cos(n+\tfrac{1}{2})x+\frac{1}{n-1/2}\cos(n-\tfrac{1}{2})x\right]_0^{\pi} \\ &= \frac{4}{\pi}\frac{-1}{4n^2-1} \end{aligned}$$

So that

$$\sin\left|\frac{x}{2}\right| = \frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{-1}{4n^2-1}\cos nx.$$

(b) Calculate

$$\sum_{n=1}^{\infty}(-1)^n/(4n^2-1).$$

Let $x = \pi$ in the answer for part (a). Then we see that

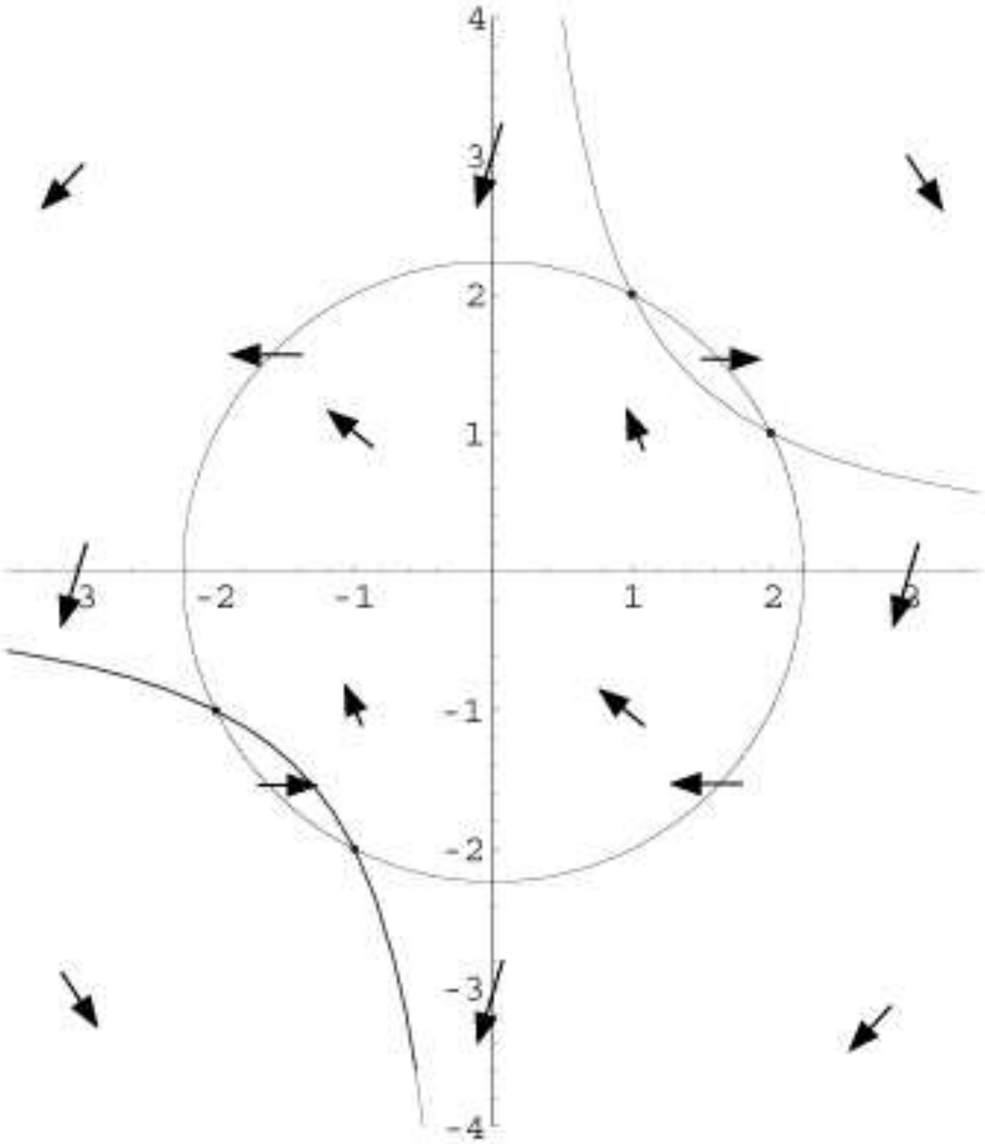
$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = -\frac{\pi}{4} + \frac{1}{2}.$$

(8) Consider the dynamical system

$$\begin{aligned} dx/dt &= 6xy - 12 \\ dy/dt &= 25 - 5x^2 - 5y^2. \end{aligned}$$

(a) On the following diagram we have marked the nullclines and equilibrium points for this dynamical system. Sketch a rough direction field.



(b) Near the equilibrium point $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ if we write $x = -2 + u$ and $y = -1 + v$ then the linearized system is

$$\begin{bmatrix} du/dt \\ dv/dt \end{bmatrix} = \begin{bmatrix} -6 & -12 \\ 20 & 10 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Describe the behavior of the original dynamical system near this equilibrium point.

The eigenvalues of the linearized system at this point are given by

$$(-6 - \lambda)(10 - \lambda + 240 = \lambda^2 - 4\lambda + 180 = 0$$
$$\Rightarrow \lambda = 2 \pm \sqrt{176} i$$

Since the eigenvalues have imaginary components, and since both have positive real part, the system spirals outward from the point $(-2, -1)$.

(c) Near the equilibrium point $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$ if we write $x = -1 + u$ and $y = -2 + v$ then the linearized system is

$$\begin{bmatrix} du/dt \\ dv/dt \end{bmatrix} = \begin{bmatrix} -12 & -6 \\ 10 & 20 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of

$$\begin{bmatrix} -12 & -6 \\ 10 & 20 \end{bmatrix}$$

and describe the behavior of the original dynamical system near this equilibrium point.

Here the eigenvalues are given by

$$(-12 - \lambda)(20 - \lambda) + 60 = \lambda^2 - 8\lambda - 180 = 0$$
$$\Rightarrow \lambda_\pm = 4 \pm \sqrt{196} = 18, -10$$

The associated eigenvectors are

$$\vec{v}_{\lambda_+} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}, \ \vec{v}_{\lambda_-} = \begin{bmatrix} -30 \\ 10 \end{bmatrix}$$

Both eigenvalues have absolute value greater than 1, so the system tends to infinity along both eigenvectors.

(d) Linearize the dynamical system near the equilibrium point $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and describe the behavior of the original system near this equilibrium point.

Letting $x = 2 + u$ and $y = 1 + v$, and taking u and v to be small (so that $u^2 = v^2 = uv = 0$), we see that

$$\frac{du}{dt} = 6(2 + u)(1 + v) - 12 \approx 6u + 12v$$

$$\frac{dv}{dt} = 25 - 5(2 + u)^2 - 5(1 + v)^2 \approx -20u - 10v$$

$$\Rightarrow \begin{bmatrix} du/dt \\ dv/dt \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ -20 & -10 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

The eigenvalues of the linearized system are:

$$(6 - \lambda)(-10 - \lambda) + 240 = \lambda^2 + 4\lambda + 180 = 0$$

$$\Rightarrow \lambda = -2 \pm \sqrt{176}i$$

Since the real part of both eigenvalues is negative, the system spirals inward near this point.

(e) Linearize the dynamical system near the equilibrium point $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and describe the behavior of the original system near this equilibrium point.

Letting $x = 1 + u$ and $y = 2 + v$ we get

$$\begin{bmatrix} du/dt \\ dv/dt \end{bmatrix} = \begin{bmatrix} 12 & 6 \\ -10 & -20 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The eigenvalues of this system are $\lambda_\pm = -4 \pm \sqrt{196} = 10, -18$, and the eigenvectors are:

$$\vec{v}_{\lambda_+} = \begin{bmatrix} 30 \\ -10 \end{bmatrix}, \; \vec{v}_{\lambda_-} = \begin{bmatrix} -2 \\ -10 \end{bmatrix}$$

Again the system tends to infinity along both eigenvectors.

(f) Which equilibrium points are stable and which are unstable?

Only the equilibrium at $(2, 1)$ is stable; as we have seen the system sprials in to this point, while all trajectories tend outward from the remaining three equilibrium points.

(g) Sketch carefully a possible phase portrait for the system.