Math 21b Exam 1 Tuesday, October 22, 1996

This exam is 90 minutes long.

1) (18 points) True or False. (Circle one) You need not give your reasoning.

a) The kernel of rref(**A**) is the same as the kernel of **A**. a) T F

b) The image of rref(**A**) is the same as the image of **A**. b) T F

c) Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be an orthonormal basis for $\mathbf{R}^n$. c) T F

Then $\mathbf{w}_1\cdot\mathbf{x} = \mathbf{w}_2\cdot\mathbf{x} = \ldots = \mathbf{w}_n\cdot\mathbf{x} = 0$ implies that $\mathbf{x} = \mathbf{0}$.

d) $\text{rank}(\mathbf{A}^2) \le \text{rank}(\mathbf{A})$ for any square matrix **A** . d) T F

e) Given two subspaces V, W of $\mathbf{R}^n$, define $V\backslash W = \{\mathbf{v} : \mathbf{v}\in V, \text{ but } \mathbf{v}\notin W\}$. V\W is a subspace. e) T F

f) Consider a system $\mathbf{Ax} = \mathbf{b}$. This system is consistent if and only if $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A}|\mathbf{b}])$, where $[\mathbf{A}|\mathbf{b}]$ denotes the augmented matrix. f) T F

2) (22 points) Let **A** be the 4 x 5 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 & 5 & 0 \\ 1 & -2 & -1 & 1 & 0 \\ 3 & -6 & -1 & 7 & -1 \\ -1 & 2 & 0 & -3 & 0 \end{bmatrix}.$$

a) Find a basis for the image of **A**.
b) Find a basis for the kernel of **A**.
c) Find a basis for $(\text{image}(\mathbf{A}))^\perp$, the orthogonal complement of image(**A**).

3) (20 points) Let *S* be a shear transformation along the *x*-axis, sending the vector $\mathbf{e}_2$ to the vector $2\mathbf{e}_1 + \mathbf{e}_2$. Let *P* be orthogonal projection onto the *x*-axis. Find matrices for *P*, *S*, and *PS*. What is

$$(PS)^{50}\begin{bmatrix} 22 \\ 17 \end{bmatrix}?$$

4) (24 points)

a) Given linear transformations T: $\mathbf{R}^7 \to \mathbf{R}^3$ and S: $\mathbf{R}^3 \to \mathbf{R}^7$, show that ker(S∘T) is at least of dimension 4. Is S∘T invertible?

b) If $\mathbf{T}^5 = \mathbf{T}^3$, show that every vector in $\text{im}(\mathbf{T}^3)$ must be in $\ker(\mathbf{T}^2 - \mathbf{I})$ and that every vector in $\text{im}(\mathbf{T}^2 - \mathbf{I})$ must be in $\ker(\mathbf{T}^3)$.

c) For what choices of the constant *k* is the following matrix invertible?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$$

d) If L and L′ are perpendicular lines in $\mathbf{R}^2$ (through the origin), what is

$$\text{proj}_L(\mathbf{x}) - \text{proj}_{L'}(\mathbf{x})$$

in simpler geometric terms?
(Here **x** denotes any vector in $\mathbf{R}^2$.)

5) (16 points) Given the linear system:

$$\begin{aligned} 2x + y - 2z &= 2 \\ -3x + 3y + 5z &= 7 \\ -x + 2y + 2z &= 1 \end{aligned}$$

a) Find the inverse of the matrix of coefficients **A**.

b) Check your answer to part (a) by calculating the product $\mathbf{A}^{-1}\mathbf{A}$.

c) Use $\mathbf{A}^{-1}$ to solve the given system of equations.