Last Name:_____

First Name:_____

Mathematics 21b

First Exam
October 21, 1998

Your Section (circle one):

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MWF 10	MWF 11	TuTh 10

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
Total	50	

No calculators are allowed.

Justify your answers carefully (except in Question 1).

Except for question 1, no credit can be given for unsubstantiated answers.

1. Are the following statements true or false?

(a)   T   F   If A is an invertible $n \times n$ matrix then the linear system $A\vec{x} = \vec{0}$ has infinitely many solutions.

(b)   T   F   There is a 2×2 matrix A with $A^{17} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(c)   T   F   If A is any 4×3 matrix and if $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are linearly dependent vectors in $\mathbf{R}^3$ then $A\vec{v}_1$, $A\vec{v}_2$ and $A\vec{v}_3$ are linearly dependent in $\mathbf{R}^4$.

(d)   T   F   Any linear transformation from $\mathbf{R}^2$ to $\mathbf{R}$ is the composite of a rotation-dilation and the linear transformation $\mathbf{R}^2 \to \mathbf{R}$ which sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to x.

(e)   T   F   A linear transformation which sends each point on the x-axis to itself is a shear.

(f)   T   F   For all real numbers c the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ c & 0 & 1 \end{bmatrix}$ is invertible.

(g)   T   F   The kernel of an invertible $n \times n$ matrix is $\{\vec{0}\}$.

(h)   T   F   There is a 2×2 matrix with the same image and kernel.

(i)   T   F   If A is an invertible $n \times n$ matrix and if B is any $n \times n$ matrix then $\ker B = \ker BA$.

(j)   T   F   If A is an $n \times n$ matrix then the rank of $A - I_n$ is greater than or equal to the dimension of $\ker A$.

1

2. For every value of k, find all solutions of the linear system

$$\begin{aligned}
x &+ ky &+ z &= 1 \\
x &+ ky &- z &= 3 \\
kx &+ y & &= 2.
\end{aligned}$$

[HINT: Consider separately the cases $k = 1$ and $k = -1$.]

3. (a) Find the matrix representing the linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ which is reflection in the line spanned by $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(b) Given that the matrix $\begin{bmatrix} 7 & -4 \\ 9 & -5 \end{bmatrix}$ represents a shear parallel to a line L in the plane, find the line L.

4. In this question A and B will always be 3×3 matrices for which AB has rank 1.

   (a) Can BA have rank 3? (Remember to justify your answer.)

   (b) Give an explicit example of two such matrices A and B for which BA has rank 2.

(c) What is the minimal possible value for the rank of BA? Give an explicit pair of such matrices A and B for which the rank of BA realises this minimum value.