Math 21b
First Midterm
March 7, 2000

1. (9 points)

(a) (6 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with $n > m$. Show that $\ker(T) \neq \{0\}$.

(b) (3 points) Find an example of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ with $\ker(T) = \{0\}$.

2. (11 points) The points $(2, 2)$, $(-1, 1)$, and $(-2, -6)$ all lie on a circle in $\mathbb{R}^2$ with equation $x^2 + y^2 + cx + dy + e = 0$.

(a) (3 points) Write the system of linear equations which will determine c,d, and e.

(b) (6 points) Write the *augmented matrix* for this system and find its *reduced row echelon form*.

(c) (2 points) Find all solutions to this system, and identify the center and radius of the circle.

3. (10 points) Let $v_1 = (1, 1)$ and $v_2 = (-1, 1)$ be two vectors in $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(v_1) = (1, 1)$ and $T(v_2) = (0, 2)$.

(a) (5 points) Find the matrix for T.

(b) (5 points) Show that T is a *shear*.

4. (11 points) Let A be a 3×3 matrix such that $A^2 = 0$. (That is, the product of A with itself is the zero matrix.)

(a) (4 points) Show that $\mathrm{Im}(A)$ is a subspace of $\ker(A)$.

(b) (4 points) Determine all *possible* values for $\mathrm{rank}(A)$, and justify your answer(s).

(c) (3 points) Give an example of such a matrix A for each possible rank.

5. (8 points) Show that the three vectors $v_1 = (2, -3, 4)$, $v_2 = (2, -5, 2)$, and $v_3 = (-4, 5, -9)$ all lie in the same plane. What does this say about their linear independence?

6. (11 points) Let $A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$.

(a) (5 points) Find a matrix B such that $B^2 = A$.

(b) (4 points) Determine $\mathrm{rank}(B)$ with justification. *(Hint: It is possible to answer part (b) without having solved part (a).)*

(c) (2 points) Determine B^{17}.

Math 21b Fall '97 Exam 1

1. True or false?
   (a) For any matrix A, $\mathrm{Im}\,(A) = \mathrm{Im}\,(\mathrm{rref}\,(A))$.
   (b) For any matrix A, $\dim\,(\mathrm{Im}\,(A)) = \mathrm{rank}\,(A)$.
   (c) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are any linearly dependent vectors in R^n, then $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.
   (d) There is a 3×6 matrix whose kernel is two-dimensional.
   (e) There is a 2×2 matrix A such that $A^2 = -I_2$.

2. Each of the spaces V_i below is equal to one (and only one) of the spaces W_j. Match the spaces.

$$V_1 = \mathrm{Im} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad W_1 = \mathrm{Im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V_2 = \mathrm{Im} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad W_2 = \mathrm{Im} \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$V_3 = \mathrm{Ker} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad W_3 = \mathrm{Ker} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$V_4 = \mathrm{Ker} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad W_4 = \mathrm{Ker} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$V_5 = \mathrm{Span} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right) \qquad W_5 = \mathrm{Span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

3. Let $A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 0 \\ 0 & 2 & 5 \end{bmatrix}$.
   (a) Is A invertible? If so, find A^{-1}.
   (b) Find A^2.

4. Let A be a 2×2 matrix (not equal to I_2) representing a shear parallel to a line L in the plane. Find
   (a) $\mathrm{Ker}\,(A - I_2)$
   (b) $\mathrm{Im}\,(A - I_2)$
   (c) $(A - I_2)^2$

5. (a) Let A be a 3×3 matrix for which $\mathrm{Im}\,(A) = \mathrm{Span} \left(\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right)$. What is $\mathrm{rank}\,(A)$? Give an example of such a matrix A.
   (b) Let B be a 3×3 matrix for which $\mathrm{Ker}\,(B) = \mathrm{Span} \left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$. What is $\mathrm{rank}\,(B)$? Give an example of such a matrix B.
   (c) Could you have chosen A and B so that $\mathrm{rank}\,(AB) = 2$? Briefly justify your answer.

Math 21b Spring '97 Exam 1

1. True or false?

   (a) If S and A are invertible $n \times n$ matrices, then
   $$(SAS^{-1})^{-1} = S^{-1}A^{-1}S.$$

   (b) The vectors
   $$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -2 \\ 3 \\ 1 \\ 6 \end{bmatrix}$$
   are linearly independent.

   (c) If the coefficient matrix of a system of m equations in n variables has rank less than m, the system has either no solutions or infinitely many solutions.

   (d) A linear transformation from R^5 to R^3 has a kernel of dimension at least 2.

   (e) If A is an $n \times n$ matrix such that $A^2 + 2A - I_n = 0$, then A is invertible.

2. Find a basis of the kernel of A and a basis of the image of A, where
$$A = \begin{bmatrix} 2 & -2 & 0 & 4 & -4 \\ -1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 2 & 2 & 1 \end{bmatrix}.$$

3. For which choices of the constant k is the matrix
$$\begin{bmatrix} k^2 & k & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
invertible?

4. Find three 2×2 matrices A, B, and C, each representing a shear parallel to either the x-axis or the y-axis, such that ABC represents a counterclockwise rotation through 90°. Draw the images of the unit square (with corners at (0,0); (1,0); (1,1); and (0,1)) under the transformations C, BC, and ABC.

5. Let $A = \begin{bmatrix} \sqrt{6} & -\sqrt{2} \\ \sqrt{2} & \sqrt{6} \end{bmatrix}$.

   (a) The linear transformation $T(\vec{x}) = A\vec{x}$ is a (check one and fill in the blanks):
   - __ rotation through an angle of _____
   - __ rotation-dilation with rotation angle _____and dilation factor _____
   - __ reflection through the line _____
   - __ orthogonal projection onto the line _____
   - __ shear parallel to the line _____

   (b) What is A^{-1}?

   (c) What is A^{10}? You may write your answer in any form that does not involve powers or products of matrices.

   (d) What is A^k, where k is a positive integer? You may write your answer in any form that does not involve powers or products of matrices.

(e) What is the smallest positive integer k such that $A^k = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$ for some numbers c_1 and c_2.

Math 21b Spring '97 Exam 1 Solutions

1. (a) False:
$$(SAS^{-1})^{-1} = \left(S^{-1}\right)^{-1} A^{-1}S^{-1} = SA^{-1}S^{-1} \neq S^{-1}A^{-1}S$$
in general.

(b) False: The third vector is twice the first plus the second.

(c) False: The system $1x_1 = 7, 2x_1 = 14$ is a counterexample.

(d) True: $\dim(\mathrm{Im}) \leq 3$ and $\dim(\mathrm{Ker}) = 5 - \dim(\mathrm{Im})$.

(e) True: The equation says $A(A+2) = I$, so $A^{-1} = A + 2$.

2. Reduce to rref and solve for Ker and identify pivot columns to get a basis for Im:

$$\text{Basis of Ker}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}. \qquad \text{Basis of Im}(A) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

3. Doing Gauss-Jordan reduces A to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & (1-k^2) \end{bmatrix}$. So A is invertible if and only if $1 - k^2 \neq 0$, which is equivalent to $k \neq \pm 1$.

4. The matrix of a shear parallel to the x-axis is of the form $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, and a shear parallel to the y-axis is of the form $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$. (Think about the transformations of $\vec{e}_1$ and $\vec{e}_2$ to determine these matrices.)
We will try to write
$$(*) \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} (**)$$
(It's not wise to use two shears of the same type consecutively, which would result in another shear of the same type.) Multiplying out and solving (*) gives $b = 1$ and $a = c = -1$. Multiplying out and solving (**) gives $b = -1$ and $a = c = 1$. Either solution is fine; we choose the first, which may be written
$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$
Graphically, C maps the original unit square to a parallelogram with vertices (0,0); (1,0); (0,1); and (-1,1). Then B maps this parallelogram to a parallelogram with vertices (0,0); (1,1); (0,1); and (-1,0). That parallelogram is mapped by A to the square with vertices (0,0); (0,1); (-1,1); and (-1,0).

5. (a) Notice A is in the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. This is a rotation/dilation by counterclockwise angle $\tan^{-1}(b/a) = \pi/6$ and factor $\sqrt{a^2+b^2} = \sqrt{8}$.

(b) A^{-1} is a rotation/dilation by a clockwise angle $\pi/6$ and factor $1/\sqrt{8}$.

(c) A^{10} is a rotation/dilation by a counterclockwise angle $10\pi/6$ and factor $\sqrt{8}^{10} = 8^5$.

(d) A^{10} is a rotation/dilation by a counterclockwise angle $k\pi/6$ and factor $\sqrt{8}^{k}$.

(e) $k = 6$ gives rotation by π and dilation by $\sqrt{8}^{3}$. The matrix for this is $\begin{bmatrix} -8^3 & 0 \\ 0 & -8^3 \end{bmatrix}$.

Math 21b Fall '97 Exam 1 Solutions

1. (a) False: $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is a counterexample.
   (b) True: $\operatorname{Im} A$ is spanned by columns of A corresponding to columns of $\operatorname{rref} A$ with leading ones.
   (c) False: $\vec{v}_1 = \vec{e}_1, \vec{v}_2 = 2\vec{e}_1, \vec{v}_3 = \vec{e}_2$ is a counterexample
   (d) False: At least 3 columns without leading ones in $\operatorname{rref} A$, so $A\vec{x} = \vec{0}$ has solutions with at least 3 free variables.
   (e) True: a 90^o rotation works.

2. $V_1 = W_4$, $V_2 = W_5$, $V_3 = W_1$,$V_4 = W_2$, $V_5 = W_3$.
   To see this, find bases of the various kernels by getting the rref's of the matrices:
   (a) $V_3 = \operatorname{Ker} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = W_1$.
   (b) $W_3 = \operatorname{Ker} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} = V_5$.
   (c) $W_4 = \operatorname{Ker} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} = V_1$.
   (d) $V_4 = \operatorname{Ker} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = W_2$.
   (e) So we must be left with $V_2 = W_5$, which is easily verified.

3. A is invertible; $A^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ -5 & 0 & 3 \\ 2 & 0 & -11 \end{bmatrix}$. The matrix $A^2 = \begin{bmatrix} -1 & 6 & 15 \\ 0 & -1 & -3 \\ -2 & 10 & 25 \end{bmatrix}$.

4. (a) $\operatorname{Ker}(A - I_2) = L$ since $A\vec{x} = \vec{x}$ for any $\vec{x}$ in L (by the definition of a shear), and so $(A - I_2)\vec{x} = \vec{0}$.
   (b) $\operatorname{Im}(A - I_2) = L$ since the definition of a shear says that $A\vec{x} - \vec{x}$ is in L for any vector $\vec{x}$. (Also since $A \neq I_2$, $\operatorname{Im}(A - I_2)$ is actually all of L.)
   (c) $(A - I_2)^2 = 0$, since for any vector $\vec{x}$, $(A - I_2)\vec{x}$ is in L by part (b), and so $(A - I_2)\left((A - I_2)\vec{x}\right)$ is $\vec{0}$ by part (a).

5. (a) $\operatorname{rank}(A) = 2$ since rank equals the dimension of the image. For example, $A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$.
   (b) $\operatorname{rank}(B) = 1$ since rank equals the dimension of the domain minus the dimension of the kernel, or $3 - 2 = 1$ here. For example, $B = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix}$.
   (c) No, $\operatorname{rank}(AB) \leq 1$ since $\operatorname{Ker}(AB)$ contains $\operatorname{Ker}(B)$, because AB kills whatever B kills. So the dimension of $\operatorname{Ker}(AB)$ is at least $\dim(\operatorname{Ker} B) = 2$, implying $\operatorname{rank}(AB) = 3 - \dim(\operatorname{Ker}(AB)) \leq 1$.

Mathematics 21b

First Midterm
March 16, 1999

1. True or False (no explanation is necessary).

T F The columns of an orthogonal matrix are always orthogonal.

T F If A and B are symmetric nxn matrices, then AB is symmetric.

T F The matrix of an orthogonal projection is an orthogonal matrix.

T F If A is a $2x7$ matrix and B is a $6x2$ matrix, then the rnak of BA is at least 5.

T F If A is a 5x5 matrix with $A^7 = 1$, then A is invertible.

T F If A is the matrix of a reflection in a plane in R^3, then $A^2 = 1$.

T F If B is the matrix of a rotation in R^2 such that $B^2 = 1$ and $B \neq 1$, then B must be $\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}$.

2. a) Let M be the matrix

$$\begin{pmatrix} 1 & 2 & 7 & 3 & 1 \\ 2 & 3 & 8 & 4 & 2 \\ 3 & 4 & 9 & 5 & 3 \\ 4 & 5 & 1 & 6 & 4 \\ 5 & 9 & 1 & 7 & 5 \end{pmatrix}$$

Is M invertible? Justify your answer briefly. (No calculation is necessary).

b) If $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ and $AB = \begin{pmatrix} 9 & 10 \\ 7 & 7 \end{pmatrix}$, find the matrix B.

3. Let A be the matrix

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

a) Find the rank of A.
b) Find the basis for $ker(A)$.
c) Find a basis for $im(A)$.
d) Use Gram-Schmidt to convert the basis you found in part c) into an orthonormal basis.

4. Consider the linear system

$$\begin{vmatrix} x + 2y + z = 3 \\ x + y - z = 2 \\ x + 2y + k^2 z = k + 4 \end{vmatrix}$$

a) For which value(s) of k does this system have a unique solution?
b) For which value(s) of k does this system have infinitely many solutions? For any such k, find all solutions. c) For which value(s) of k is this system inconsistent.

5. a) Suppose A is the matrix of a rotation about a line L in R^3. If v is a vector in the direction of L, show that $ker(A - 1_3)$ is spanned by v. You may assume that the angle of rotation is non-zero.
b) Let B be the matrix for rotation rhrough 90^0 clockwise about the z-axis (clockwise as seen when facing the origin from the positive z-axis). Find B and C.
c) Find BC.
d) It is a fact that the composition of two rotations in R^3 is an other rotation. Find a vector v which spans the axis of the rotation corresponding to BC. (Hint: Use part a).