Throughout this solution set, unless otherwise stated, large **boldface** uppercase letter signify a matrix, small **boldface** letters signify a vector, small *italics* letters signify vector components and small Greek letters (αβγ) signify scalar.

1)

a) True. The kernel of **A** is the vector space spanned by the solutions to **Ax=0**, which is the same as the solution to **Bx=0**, **B**=rref(**A**).

b) False. Image(**A**) is the column span of **A**, but we obtain rref(**A**) by elementary row operations that do not preserve the properties (e.g. linear independence) of the columns.

c) True. **AAAB = AABA = ABAA = BAAA.**

d) True. Image(**A**) = Kernel(**A**) implies **A**(**Ay**) = **0** for any **y**. This means $\mathbf{A}^2 = 0$. So **A** could be e.g. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. Note that we could also construct the matrix directly, by creating a transformation that sends everything to some (**zw** in this case) plane but wipes out everything in that plane.

e) False. All spaces contain **0** by definition. So if Kernel(**A**)={**0**}, then it is contained in the Image(**B**) but both **A** and **B** can be invertible.

f) False. If **A** is invertible then its dimension must be equal to its rank, so rank(**A**)=2.

2) a) We want to solve **Ax**=0, so we first row reduce **A** to get:

$$\begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives us the equations:

$x_1 - 2x_2 + 3x_4 = 0 \quad \Rightarrow x_1 = 2\beta - 3\alpha$

$x_2 = \beta$

$x_3 + 2x_4 = 0 \quad \Rightarrow x_3 = -2\alpha$

$x_4 = \alpha$

$x_5 = 0$

yielding solutions of the form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ so Kernel(A) = span } \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

b) We know that the Image of **A** is the column span on **A**. From part a), we know that we need three columns, and that they must be the ones with the leading 1, so the Image(**A**) =

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$$

c) Just row reduce the augmented matrix to get:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

representing

$$\begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 12 \\ 0 \end{bmatrix}$$

yielding the equations:

$x_5 = 12$

$x_4 = \alpha$

$x_3 + 2x_4 = -1 \qquad \Rightarrow x_3 = -1 - 2\alpha$

$x_2 = \beta$

$x_1 - 2x_2 + 3x_4 = 4 \qquad \Rightarrow x_1 = 4 + 2\beta - 3\alpha$

3)

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{PS} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

note that **PS*PS=PS**, so $\mathbf{PS}^{50} = \mathbf{PS}$ so $(PS)^{50}\begin{bmatrix} 22 \\ 17 \end{bmatrix} = \begin{bmatrix} 56 \\ 0 \end{bmatrix}$

4) a)

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -2 \\ -3 & 3 & 5 \\ -1 & 2 & 2 \end{bmatrix}$$ so consider the augmented matrix: $$\left[\begin{array}{ccc|ccc} 2 & 1 & -2 & 1 & 0 & 0 \\ -3 & 3 & 5 & 0 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

and row reduce it to get: $\mathbf{A}^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & 6 & -11 \\ 0 & 1 & 0 & - & -2 & 4 \\ 0 & 0 & 1 & & 5 & -9 \end{array}\right]$

b) Just multiply it out to get $\mathbf{AA}^{-1}=\mathbf{I}$

c) using $\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$ we get $\mathbf{x} = \begin{matrix} 4 & 6 & -11 \\ -1 & -2 & 4 \\ 3 & 5 & -9 \end{bmatrix}\begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 39 \\ -12 \\ 32 \end{bmatrix}$

5) Spot the linear system:
let s_0, m_0, l_0 be the numbers of small, medium and large cars respectively in 1990.
If s_1, m_1, l_1 are the numbers of small, medium and large cars respectively in 2000 then:

$s_1 = .5\, s_0 + .3\, m_0$
$m_1 = .4\, s_0 + .6\, m_0 + .4\, l_0$
$l_1 = .1\, s_0 + .1\, m_0 + .6\, l_0$

but since the number of cars did not change, this is

$s_0 = .5\, s_0 + .3\, m_0 \quad \Rightarrow -.5\, s_0 + .3\, m_0 = 0$
$m_0 = .4\, s_0 + .6\, m_0 + .4\, l_0 \quad \Rightarrow .4\, s_0 - .4\, m_0 + .4\, l_0 = 0$
$l_0 = .1\, s_0 + .1\, m_0 + .6\, l_0 \quad \Rightarrow .1\, s_0 + .1\, m_0 - .4\, l_0 = 0$

row reduce:

$$\begin{bmatrix} 1 & 0 & -1.5 \\ 0 & 1 & -2.5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} s_0 \\ m_0 \\ l_0 \end{bmatrix} = \mathbf{0}$$

yielding

$l_0 = t$
$m_0 = 2.5t$
$s_0 = 1.5t$

the survey was conducted among 1000 drivers, so $t+2.5t+1.5t = 1000 \Rightarrow 5t = 1000 \Rightarrow t=200$
so $l_0 = 200$, $m_0 = 500$, $s_0 =300$.

6) a) To compute the matrix of the transformation, all we need to know is what the projection does to the basis vectors. Using the formula for orthogonal projection onto a line:

$$\vec{e}_i^{\,\prime} = \vec{l}\,\frac{\vec{e}_i . \vec{l}}{|l|^2}$$

we get the matrix:

$$\mathbf{A} = \frac{1}{9}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}$$

b) We could compute the kernel by solving **Ax=0**, but there is an easier way. Since **A** is a matrix associated with an orthogonal projection onto L, only lines perpendicular to L will be reduced to 0. From dot product properties:

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = 0$$

we get $s_1 + 2s_2 - 2s_3 = 0$

set s_1=0 we get $\mathbf{k}_1$=[0,1,-1]

set s_2=0 we get $\mathbf{k}_2$=[2,0,1]

set s_3=0 we get $\mathbf{k}_3$=[-2,1,0]

but note that $\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$

So Kernel(**A**) = span {[2,0,1],[-2,1,0]}

c) First note that we have calculated **p,** or rather the matrix that transforms **x** into **p** in part a) of this problem. So we can write **p=Ax.** Let **B** be the matrix of the orthogonal projection onto the plane S. Then, from the hint:

$\mathbf{Bx} = \mathbf{x} - \mathbf{p} = \mathbf{x} - \mathbf{Ax} = \mathbf{Ix} - \mathbf{Ax} = (\mathbf{I}\text{-}\mathbf{A})\mathbf{x}$

$$\text{So } \mathbf{B} = \mathbf{I} - \mathbf{A} = \frac{1}{9}\begin{bmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$