Name: *Answer Sheet*

Math 21b Midterm 1 Thursday, October 24th, 2002

Please circle your section:

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MWF 9-10	MWF 10-11	MWF 11-12

Question	Points	Score
1	18	
2	15	
3	18	
4	12	
5	15	
6	12	
7	10	
Total	100	

You have two hours to take this midterm. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You don't have to answer the problems in any particular order. So move on to another problem if you find you're stuck and that you are spending too much time on one problem.

To receive full credit on a problem, you will need to justify your answers carefully - unsubstantiated answers, even if correct, will receive little or no credit (except if the directions for that question specifically say no justification is necessary, such as the True/False).

Please be sure to write neatly - illegible answers will also receive little or no credit.

If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of that so that the grader knows where to find your answers.

You are allowed one 3 by 5 inch file card with formulas on it during the test, but you are not allowed to use any other notes, or calculators during this test.

Good luck! Focus and do well!

Question 1. (18 points total)

True or False (3 points each) No justification is necessary, simply circle **T** or **F** for each statement.

T **F** (a) If A is invertible then the number of rows of A must equal the rank of A.

A invertible means A is $n \times n$ and $\text{rref}(A) = I_n$,
so $\text{rank}(A)$ must equal $n = \#$ of rows of A

T **F** (b) If A has a nonzero kernel, then A is not invertible.

Again, A invertible means $\text{rref}(A) = I_n$,
and so A invertible implies $\ker(A) = \{\vec{0}\}$,
so if A has nonzero kernel, then A isn't invertible

T **F** (c) The only 2 x 2 matrix A with $A^2 = I_2$ is the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

No, try simply $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

T **F** (d) Let A and B be two n x n matrices such that $\ker(B) \subseteq \ker(A)$. Then the matrix AB is not invertible.

No, again try a simple counterexample: $A = B = I_n$
Then $\ker(B) = \{\vec{0}\} \subseteq \ker(A) = \{\vec{0}\}$, but of
course $AB = I_n$ and $(AB)^{-1} = I_n$, i.e. it is invertible

T **F** (e) If A is a 5 x 4 matrix of rank 4 then the kernel of A must be nonzero.

No, look at $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\ker(A) = \{\vec{0}\}$
$\text{rank}(A) = 4$

T **F** (f) Let A and B be two matrices such that the product AB is defined. If B has a column of zeros, then AB must also have a column of zeros.

if $B = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ then $AB = \begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_n \end{bmatrix}$
so if one of B's columns, say $\vec{v}_i = \vec{0}$,
then $A\vec{v}_i = \vec{0}$, so AB has a column of zeros

Question 2 (15 points total)

Suppose that $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are matrices for two linear transformations.

(a) (4 points) Briefly describe the two transformations given by A and B geometrically

A sends $\vec{e}_1$ to $-\vec{e}_2$, and $\vec{e}_2$ to $\vec{e}_1$:

thus A is a $90°$ clockwise rotation

(or check $A = \begin{bmatrix} \cos 90° & \sin 90° \\ -\sin 90° & \cos 90° \end{bmatrix}$)
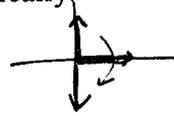
B collapses the x-axis down to $\vec{0}$ (i.e. kernel (B) contains x-axis)

and preserves the y-axis, $B(\vec{e}_2) = \vec{e}_2$, so B is the

orthogonal projection onto the y-axis

(b) (4 points) What are the ranks of A and B? Is either A or B invertible? If not, justify your answer, if so then find its inverse.

$\text{rref}(A) = \mathbb{I}_2$, so rank $(A) = 2$, while $\text{rref}(B) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

with rank $(B) = 1$.

As rank $(B) = 1 < 2$, B is not invertible.

$A^{-1} = \dfrac{1}{\det(A)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(c) (4 points) Do A and B commute? Justify your answer by describing the transformations AB and BA geometrically.

AB represents the transformation that projects onto the y-axis, then rotates $90°$ clockwise, thus the image of AB equals the x-axis (that is the y-axis after a $90°$ rotation).

However BA represents a $90°$ rotation of the plane followed by projection onto the y-axis, so the image of BA is the y-axis. As the image $(AB) \ne$ image (BA) these two transformations are different, so A and B do not commute, i.e. $AB \ne BA$

(d) (3 points) What is A^{2002}?

Since A represents a $90°$ (clockwise) rotation then A^4 rotates $360°$, i.e. $A^4 = \mathbb{I}_2$. Then $A^{2002} = A^{2000} A^2$

and $A^{2000} = (A^4)^{500} = (\mathbb{I}_2)^{500} = \mathbb{I}_2$, so $A^{2002} = A^2$

$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Question 3. (18 points total)

(a) (10 points) Find all solutions to the following homogeneous linear system (if there is more than one solution, express the solutions in vector form, i.e. as a linear combination of column vectors).

$$\begin{cases} 2x_1 - x_2 + 3x_3 + 3x_4 = 0 \\ -2x_1 \quad\quad + x_3 + 4x_4 = 0 \\ 10x_1 - 3x_2 + 7x_3 + x_4 = 0 \end{cases}$$

Crank out the rref! start with $\begin{bmatrix} 2 & -1 & 3 & 3 \\ -2 & 0 & 1 & 4 \\ 10 & -3 & 7 & 1 \end{bmatrix} \begin{matrix} +(I) \\ \\ -5(I) \end{matrix}$

(don't need to keep writing the extra column of 0's at the right)

Also - we might take a few shortcuts to avoid fractions where possible...

$\rightarrow \begin{bmatrix} 2 & -1 & 3 & 3 \\ 0 & -1 & 4 & 7 \\ 0 & 2 & -8 & -14 \end{bmatrix} \div (-1) \quad \begin{bmatrix} 2 & -1 & 3 & 3 \\ 0 & 1 & -4 & -7 \\ 0 & 2 & -8 & -14 \end{bmatrix} \begin{matrix} +(II) \\ \\ -2(II) \end{matrix} \rightarrow \begin{bmatrix} 2 & 0 & -1 & -4 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

okay, finally $\div$ first row by 2: $\begin{bmatrix} 1 & 0 & -\frac{1}{2} & -2 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ done...

So $\quad x_1 = \frac{1}{2}x_3 + 2x_4$

$x_2 = 4x_3 + 7x_4 \quad$ solutions: $\quad s\begin{bmatrix} \frac{1}{2} \\ 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 7 \\ 0 \\ 1 \end{bmatrix}$

$x_3 = s$

$x_4 = t$

$\left(\text{or} \quad s\begin{bmatrix} 1 \\ 8 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 7 \\ 0 \\ 1 \end{bmatrix} \quad \text{if you don't like fractions!}\right)$

Note: check $\begin{bmatrix} 1 \\ 8 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 7 \\ 0 \\ 1 \end{bmatrix}$ in the original equations to make sure they're both solutions!

(b) (3 points) Is it possible to give a nontrivial solution to the linear system in part (a) with $x_3 = x_4 = 0$? If so, write down one such solution, if not justify your answer

So according to the solution in part (a) if $x_3 = x_4 = 0$, then in the solution as written this corresponds to $x_3 = s = 0$, $x_4 = t = 0$, so the only solution with $x_3 = x_4 = 0$ is $0\begin{bmatrix} 1 \\ 8 \\ 2 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \vec{0} \rightarrow$ i.e. it's not possible to give a nontrivial solution in this case.

$\left(\text{or. note } x_3 = x_4 = 0 \text{ implies } x_1 = 0 \text{ from equation (II) in the original system, this then implies } x_2 = 0 \text{ from either (I) or (III)}\right)$

(c) (5 points) Find all values for α for which the following homogeneous linear system has nontrivial solutions.

$$\begin{cases} 2x_1 + x_3 = 0 \\ -\alpha x_2 + x_3 = 0 \\ (1-\alpha^2)x_3 = 0 \end{cases}$$

Again, try to work out the rref of

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -\alpha & 1 \\ 0 & 0 & (1-\alpha^2) \end{bmatrix} \begin{matrix} \div 2 \\ \div (-\alpha) \\ \ \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{\alpha} \\ 0 & 0 & (1-\alpha^2) \end{bmatrix} \div (1-\alpha^2) \rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{\alpha} \\ 0 & 0 & 1 \end{bmatrix}$$

note, can't do this if $\alpha = 0$, so need to check $\alpha = 0$ separately...

can't do this if $(1-\alpha^2) = 0$ so check when this occurs.

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The original matrix will have only the trivial solution when the rref $= I_3$, which happens when $\alpha \neq 0$ and $(1-\alpha^2) \neq 0$, or $\alpha \neq \pm 1$.

Now check: when $\alpha = 0$ the system becomes

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ so solution is anything of the form $\begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix}$, $s \in \mathbb{R}$

$\rightarrow$ non trivial solution

when $\alpha =$ either $+1$ or -1 we get

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & \pm 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

with solutions either $x_1 = -\frac{1}{2}x_3$

$x_2 = \pm x_3$

$x_3 = s$

So either $s\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$ or $s\begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$

either way $\rightarrow$ nontrivial solutions

So the system has nontrivial solutions only when $\alpha = 0, -1$ or $+1$

Question 4. (12 points total)

Suppose you know that the inverses of two 3 x 3 matrices, A and B, are given as follows:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & 1 \\ 2 & -2 & 0 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

(a) (4 points) Find all solutions to $B\vec{x} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$.

$$\text{since } B\vec{x} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \quad \text{then} \quad B^{-1}B\vec{x} = B^{-1}\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ -8 \\ 0 \end{bmatrix}$$
$$\qquad\qquad\qquad \hookrightarrow = \vec{x}$$

$$\text{so } \quad \vec{x} = \begin{bmatrix} -4 \\ -8 \\ 0 \end{bmatrix} \quad \text{is the only solution.}$$

(b) (4 points) Find all solutions to $A\vec{x} = \vec{0}$

$$\text{again if } \quad A\vec{x} = \vec{0}, \quad \text{then} \quad \vec{x} = A^{-1}(A\vec{x}) = A^{-1}(\vec{0}) = \vec{0}$$
$$\text{so } \quad \vec{x} = \vec{0} \quad \text{is the only solution}$$

(c) (4 points) Consider the transformation $T(\vec{x}) = (BA)\vec{x}$, where A and B are the same two matrices as in parts (a) and (b). Find the matrix for the inverse transformation, $T^{-1}(\vec{x})$

$$\text{we know } \quad (BA)^{-1} = A^{-1}B^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & -3 & 1 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 15 \\ 0 & -12 & 6 \\ 2 & -4 & 6 \end{bmatrix}$$

Question 5. (15 points total)

Find examples of linear transformations that have the following features (i.e. write down a possible matrix for each of the transformations):

(a) (3 points) Find an example of a transformation $T : \mathfrak{R}^3 \to \mathfrak{R}^4$ that has a nonzero kernel.

Strategy: find the simplest such T possible,
we know $A = 4 \times 3$ matrix. if $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ then
all of $\mathbb{R}^3$ is in the kernel as
$A\vec{x} = 0$ for all $\vec{x}$ in $\mathbb{R}^3$

(b) (3 points) Find an example of a transformation $T : \mathfrak{R}^3 \to \mathfrak{R}^4$ that has image equal to the line

spanned by $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ Since image $(T) =$ span of columns of matrix A, then we just need the columns of A to be multiples of $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, such as $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \end{bmatrix}$!

(c) (3 points) Find an example of a transformation $T : \mathfrak{R}^3 \to \mathfrak{R}^4$ that has a kernel equal to its domain.

This is a bit silly → we've already found a trivial example of this in part (a), i.e.
$\ker(A) = \mathbb{R}^3 = $ domain (T) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) (3 points) Is it possible to find an example of a transformation $T : \mathfrak{R}^3 \to \mathfrak{R}^4$ that has an image equal to its codomain? If so, give an example, if not, explain why not.

No, as image $(T) =$ span of columns of A, which has just
3 columns, and $\mathbb{R}^4$, 4 dimensional space, can't be spanned
by just 3 vectors (or dim $(\text{Im}(A)) = $ rank $(A) \leq 3$, so
$\text{Im}(A)$ can't be $\mathbb{R}^4$)

(e) (3 points) Is it possible to find an example of a transformation $T : \mathfrak{R}^3 \to \mathfrak{R}^4$ so that $T^2 = I_3$. If so, give an example, if not, explain why not.

No, the dimensions are off! $T : \mathbb{R}^3 \to \mathbb{R}^4$
so $T(\vec{x}) \in \mathbb{R}^4$, now we can't apply T again, as T's
input has to be in $\mathbb{R}^3$. Or by dimension of matrices,
we know $T(\vec{x}) = A\vec{x}$, where A is 4×3, so A^2 does not exist.

Question 6. (12 points total)

(a) (8 points) Find all values of α for which the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & \alpha \end{bmatrix}$ exists and write down the matrix for A^{-1}.

Find rref

$$\begin{bmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 1 & 0 & 0 & | & 0 & 1 & 0 \\ 1 & 2 & \alpha & | & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ -(I) \\ -(I) \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & -1 & 0 & | & -1 & 1 & 0 \\ 0 & 1 & \alpha & | & -1 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \div(-1) \\ +(II) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 1 & -1 & 0 \\ 0 & 0 & \alpha & | & -2 & 1 & 1 \end{bmatrix} \begin{matrix} -(II) \\ \\ \div(\alpha) \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & 1 & -1 & 0 \\ 0 & 0 & 1 & | & -\frac{2}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} \end{bmatrix}$$

can't do if $\alpha = 0$

So $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -\frac{2}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} \end{bmatrix}$ for $\alpha \neq 0$ $\left(\text{i.e. } A^{-1} \text{ exists for all } \alpha \in \mathbb{R} \text{ except } \alpha = 0\right)$

$\left(\text{again, can check that } AA^{-1} = I_3\right)$

(b) (4 points) Suppose α is chosen to be equal to 1 (so that the matrix A^{-1} does in fact exist) then calculate the matrix product $\left(A^{-1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} A\right)^2$

Note $\hookrightarrow = \left(A^{-1}(2I_3)A\right)^2 = \left(2 A^{-1}I_3 A\right)^2$

$$= \left(2 A^{-1}A\right)^2 = \left(2 I_3\right)^2 = 2^2 I_3$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Question 7. (10 points total)

Let A be an n x n invertible matrix and let B be an n x r matrix. Show that $\ker(AB) = \ker(B)$
Hint: First show that $\ker(B) \subseteq \ker(AB)$ by taking a vector in $\ker(B)$ and show that this vector must also be in $\ker(AB)$. Then show that $\ker(AB) \subseteq \ker(B)$

Following the hint:

Suppose $\vec{x} \in \ker(B)$, then this means $B\vec{x} = \vec{0}$

So $(AB)\vec{x} = A(B\vec{x}) = A(\vec{0}) = \vec{0}$,

So $\vec{x}$ is also in $\ker(AB)$,

This means $\ker(B) \subseteq \ker(AB)$

Next, suppose $\vec{x} \in \ker(AB)$, then $AB\vec{x} = \vec{0}$.

Now since A is invertible, then A^{-1} exists,

So $B\vec{x} = A^{-1}(AB\vec{x})$ i.e. $A^{-1}A = I_n$

$= A^{-1}(\vec{0}) = \vec{0}$

So $\vec{x}$ is also in $\ker(B)$ since $B\vec{x} = \vec{0}$

Thus $\ker(AB) \subseteq \ker(B)$

Since $\ker(B) \subseteq \ker(AB)$

and $\ker(AB) \subseteq \ker(B)$

then in fact, $\ker(B)$ must equal $\ker(AB)$