Last Name: ______________________________

First Name: ______________________________

Math 21b
Second Midterm
April 11, 2000

Please circle your section:

Christine Taylor	Alex Popa	Bo Cui	Dmitry Tamarkin	John Boller	John Boller
10 MWF	11 MWF	11 MWF	12 MWF	10 TTh	11:30 TTh

Problem	Points	Score
1	11	
2	9	
3	6	
4	12	
5	10	
6	12	
Total	60	

1. (11 points) Let M_n be the $n \times n$ matrix with 0's along the diagonal and 1's everywhere else. For example, $M_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$.

(a) (2 points) Find $\det(M_2)$ and $\det(M_3)$.

(b) (5 points) Find $\det(M_n)$ for every n and justify your answer. *(Hint: Put M_n in upper-triangular form.)*

(c) (4 points) Find all eigenvalues and their algebraic multiplicities.

2. (9 points) We say that a matrix A is **skew-symmetric** if $A = -A^T$. In all of the following, assume that A is a real $n \times n$ skew-symmetric matrix.

(a) (2 points) Find $\text{tr}(A)$.

(b) (3 points) Find $\det(A)$ in the case where n is odd.

(c) (4 points) Show that if λ is an eigenvalue of A, then $-\lambda$ is also an eigenvalue of A.

3. (6 points) Let A be an $n \times n$ matrix such that $A^2 = 2A - I$. What are the possible eigenvalues of A?

4. (12 points) Let A be a real $n \times n$ matrix such that $A^4 = -I$.

(a) (4 points) Show that n must be even.

(b) (4 points) One student found an eigenvalue of $1+i$. Why is this answer incorrect?

(c) (4 points) Another student found an eigenvalue of $\frac{1+i}{\sqrt{2}}$ with a corresponding eigenvector $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 6 \end{bmatrix}$. Why is this answer incorrect?

5. (10 points) Let $\mathbf{w_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{w_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{w_3} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

(a) (5 points) Construct an orthonormal basis for the three-dimensional subspace of $\mathbb{R}^4$ spanned by $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$, and call the vectors of your basis $\mathbf{v_1}$,$\mathbf{v_2}$, and $\mathbf{v_3}$.

(b) (5 points) Let $A = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \end{bmatrix}$. Compute $\det(AA^T)$ and $\det(A^T A)$ without multiplying out these matrices.

6. (12 points) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 4 \end{bmatrix}$. You are given that 4 is an eigenvalue for A.

(a) (3 points) Find all eigenvalues for A.

(b) (5 points) Does A have an eigenbasis? If so, find one.

(c) (4 points) Evaluate $\lim_{n\to\infty} A^n \mathbf{e_1}$ where $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. *(Hint: It is possible to write* $\mathbf{e_1}$ *as a linear combination of two of the eigenvectors.)*