Last Name:_____

First Name:_____

Mathematics 21b

Second Exam
April 10, 2001

Your Section (circle one):

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Question	Score
1	
2	
3	
4	
5	
Total	

The exam will last 90 minutes.

The exam consists of 5 questions, the first worth 12 points, the second worth 8 points and the others each worth 10 points.

No calculators are allowed.

Justify your answers carefully (except in Questions 1 and 2).

For Questions 3–5, no credit can be given for unsubstantiated answers.

Write your final answers in the spaces provided.

(1) True or False (no explanation is necessary).

T F : If V is a subspace of $\mathbf{R}^n$ and $\vec{x}, \vec{y} \in \mathbf{R}^n$ then $||\operatorname{proj}_V \vec{x} - \operatorname{proj}_V \vec{y}|| \leq ||\vec{x} - \vec{y}||$.

T F : All shears are orthogonal transformations.

T F : If a 2×2 orthogonal matrix A has exactly one eigenvalue then $A = \pm I_2$.

T F : If a matrix A has QR-factorisation QR then $\ker A = \ker R$.

T F : If all the columns of an $n \times n$-matrix A are unit vectors then $|\det A| \leq 1$.

T F : If an $n \times n$-matrix A has eigenvalue 2 and an $n \times n$-matrix B has eigenvalue 5, then AB has eigenvalue 10.

(2) For each of the following values of d state whether or not $\vec{0}$ is an (asymptotically) stable equilibrium of the dynamical system

$$\vec{x}(t+1) = \begin{pmatrix} 1 & -1 \\ d & 0 \end{pmatrix} \vec{x}(t).$$

[Circle **S** for "stable" or **U** for "unstable".]

S **U** : $d = 2$.

S **U** : $d = 1$.

S **U** : $d = 1/2$.

S **U** : $d = 1/8$.

[This page is blank for rough working.]

(3) Let $V \subset \mathbf{R}^4$ denote the image of

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

(a) Find the matrix of the orthogonal projection onto V.

(b) Find the matrix of orthogonal projection onto $V^{\perp}$. [HINT: Consider $\text{proj}_V + \text{proj}_{V^{\perp}}$.]

(4) (a) Find the least squares solution for the equations

$$\begin{array}{rcrcl} x & + & y & = & 4 \\ & & y & = & 2 \\ x & & & = & -1. \end{array}$$

(b) Calculate the determinant of

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{pmatrix}.$$

(c) Calculate the determinant of

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

(5) Let A be a 3×5 matrix and B a 5×3 matrix with $AB = I_3$.

(a) Explain why A has rank 3 and why B has rank 3.

(b) Show that the non-zero vectors in $\ker A$ are eigenvectors of BA. What are the corresponding eigenvalues?

(c) Show that the non-zero vectors in $\operatorname{Im} B$ are eigenvectors of BA. What are the corresponding eigenvalues?

(d) Find the characteristic polynomial of BA.