

## Math 21b Midterm 2 Solutions - Spring 2001

1. True or False (no explanation is necessary):

- TRUE - Since  $\text{proj}_V \vec{v}$  and  $\vec{v} - \text{proj}_V \vec{v}$  are the legs of the right triangle with hypotenuse  $\vec{v}$ ,  $\|\text{proj}_V \vec{v}\| \leq \|\vec{v}\|$  for all  $\vec{v} \in \mathbb{R}^n$ . Also, orthogonal projection is a linear transformation, so we have  $\|\text{proj}_V \vec{x} - \text{proj}_V \vec{y}\| = \|\text{proj}_V(\vec{x} - \vec{y})\| \leq \|\vec{x} - \vec{y}\|$ .
- FALSE - Consider the shear defined by the matrix:

$$\begin{bmatrix} 1 & .5 \\ 0 & 1 \end{bmatrix}.$$

This sends the unit vector  $(0, 1)$  to the vector  $(\frac{1}{2}, 1)$ , which has length  $\frac{\sqrt{5}}{2}$ . Therefore this shear does not preserve length so it is not an orthogonal transformation.

- TRUE - A real orthogonal matrix has eigenvalues  $\pm 1$ . If both eigenvalues are 1, then the matrix is the identity. If both eigenvalues are  $-1$ , then the matrix must be  $-I_2$ .
- TRUE - To have a QR-factorization, the  $n \times m$  matrix  $A$  must have linearly independent columns. Could  $\ker(A) \neq \vec{0}$ ? Suppose for  $\vec{v} = (v_1, \dots, v_m)$ ,  $A\vec{v} = \vec{0}$  where the columns of  $A$  are  $\vec{w}_1, \dots, \vec{w}_m$ . Then  $v_1\vec{w}_1 + \dots + v_m\vec{w}_m = \vec{0}$ . Since the  $\vec{w}_i$ 's are linearly independent, this means  $v_1 = \dots = v_m = 0$ . Therefore  $\ker(A) = \vec{0}$ .  $R$  is a square upper triangular matrix with positive diagonal entries. So  $\det(R)$  is the product of the diagonal entries, which is nonzero. Therefore  $\ker(R) = \vec{0}$ .
- TRUE -  $A$  is a square matrix, so the volume of the parallelepiped defined by the column vectors (call them  $\vec{v}_1, \dots, \vec{v}_n$ ) of  $A$  is  $|\det(A)|$ . This parallelepiped has volume at most  $\prod_{i=1}^n \|\vec{v}_i\| = \prod_{i=1}^n 1 = 1$ . Therefore  $|\det(A)| \leq 1$ .
- FALSE - Consider the following matrices:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 5 & 0 \\ 5 & 5 \end{bmatrix}.$$

$A$  has eigenvalue 2 with multiplicity 2, and  $B$  has eigenvalue 5 with multiplicity 2.

However, their product  $AB = \begin{bmatrix} 15 & 5 \\ 10 & 10 \end{bmatrix}$  has eigenvalues 5 and 20.

2. Determine if  $\vec{0}$  is a stable equilibrium:

- UNSTABLE - The characteristic polynomial when  $d = 2$  is  $\lambda^2 - \lambda + 2 = 0$ . The roots of this equation are  $\lambda_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{7}}{2}$ .  $|\lambda_{1,2}| = \sqrt{2} > 1$ , so  $\vec{0}$  is not a stable equilibrium.

- UNSTABLE - The characteristic polynomial when  $d = 1$  is  $\lambda^2 - \lambda + 1 = 0$ . The roots of this equation are  $\lambda_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ .  $|\lambda_{1,2}| = 1$ , so the trajectory is an ellipse and  $\vec{0}$  is not a stable equilibrium.
- STABLE - The characteristic polynomial when  $d = \frac{1}{2}$  is  $\lambda^2 - \lambda + \frac{1}{2} = 0$ . The roots of this equation are  $\lambda_{1,2} = \frac{1}{2} \pm i\frac{1}{2}$ .  $|\lambda_{1,2}| = \frac{1}{\sqrt{2}} < 1$ , so  $\vec{0}$  is a stable equilibrium state.
- STABLE - The characteristic polynomial when  $d = \frac{1}{8}$  is  $\lambda^2 - \lambda + \frac{1}{8} = 0$ . The roots of this equation are  $\lambda_{1,2} = \frac{1}{2} \pm \frac{1}{2\sqrt{2}}$ .  $|\lambda_{1,2}| < 1$ , so  $\vec{0}$  is a stable equilibrium state.

3. (a)  $V$  is the image of  $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ , so  $V$  is spanned by  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ . To find the matrix of the orthogonal projection onto  $V$ , we need to find an orthonormal basis for  $V$ :

$$\vec{w}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} .5 \\ .5 \\ .5 \\ .5 \end{bmatrix}.$$

$$\vec{w}_2 = \frac{\vec{v}_2 - \text{proj}_{\vec{w}_1} \vec{v}_2}{\|\vec{v}_2 - \text{proj}_{\vec{w}_1} \vec{v}_2\|}. \quad \vec{v}_2 - \text{proj}_{\vec{w}_1} \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} .5 \\ .5 \\ .5 \\ .5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}. \quad \text{So } \vec{w}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}.$$

Therefore the matrix of the orthogonal projection onto  $V$  is:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & 0 \end{bmatrix}^T = \begin{bmatrix} 3/4 & 1/4 & -1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ -1/4 & 1/4 & 3/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

- (b) We know that  $\text{proj}_V \vec{v} + \text{proj}_{V^\perp} \vec{v} = \vec{v}$ , so the matrix for the orthogonal projection onto  $V^\perp$  plus the matrix from (a) is the identity matrix. Therefore the matrix for the orthogonal projection onto  $V^\perp$  is:

$$\begin{bmatrix} 1/4 & 3/4 & 5/4 & 3/4 \\ 3/4 & 3/4 & 3/4 & 3/4 \\ 5/4 & 3/4 & 1/4 & 3/4 \\ 3/4 & 3/4 & 3/4 & 3/4 \end{bmatrix}.$$

4. (a) Recall that the least squares solutions to the system  $A\vec{x} = \vec{b}$  are the exact solutions to the system  $A^T A\vec{x} = A^T \vec{b}$ . In this problem,  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$  and  $\vec{b} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ . So if  $\vec{x}^*$  is the least squares solution,

$$\begin{aligned} \vec{x}^* &= \left( \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}. \end{aligned}$$

- (b) Recall two facts about determinants: adding a multiple of a row to another row does not change the determinant of the matrix, and the determinant of a triangular matrix is the product of the diagonal entries. Subtract the first row of the given matrix from each of the other four rows. This will not change the determinant, so the determinant of the given matrix is the same as the determinant of:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

This matrix has determinant equal to  $(1)(1)(2)(3)(4) = 24$ .

- (c) Given the number of zeros in this matrix, there are not many nonzero patterns. Let  $a_{ij}$  indicate the entry in row  $i$  and column  $j$  of the matrix. Every nonzero pattern must use  $a_{21}$  and  $a_{34}$ . Once a pattern contains these two entries, there are not many other options. If the pattern uses  $a_{12}$  then it is either the pattern  $a_{12}a_{21}a_{34}a_{43}a_{55}$  or  $a_{12}a_{21}a_{34}a_{45}a_{53}$ . The first of these has a total of 2 inversions, and the second has 3. Therefore they contribute  $+1$  and  $-1$ , respectively, to the determinant. The only remaining nonzero patterns, then, must include  $a_{42}$ . This forces  $a_{55}$  to be in the pattern, and so  $a_{13}$  is as well. This pattern contains 3 inversions, so it contributes  $-1$  to the determinant. Thus the determinant of this matrix is  $0 + 1 - 1 - 1 = -1$ .
5. (a) If  $\text{rank}(B) < 3$ , then  $\ker(B) \neq \vec{0}$ . Say  $B\vec{v} = \vec{0}$  and  $\vec{v} \neq \vec{0}$ . Then  $\vec{0} = AB\vec{v} = I\vec{v} = \vec{v}$  - contradiction. Thus  $\text{rank}(B) = 3$  (it cannot exceed 3 since it is a  $5 \times 3$  matrix). If  $\text{rank}(A) < 3$ , then the dimension of the image of  $A$  is less than 3. But  $AB = I_3$  and the dimension of the image of  $I_3$  is 3. Thus  $\text{rank}(A) = 3$ .
- (b) Suppose  $\vec{v} \in \ker(A)$  is a nonzero vector. Then  $BA\vec{v} = B\vec{0} = \vec{0}$ .  $\vec{v} \neq \vec{0}$  so  $\vec{v}$  is an eigenvector, and its eigenvalue is 0.
- (c) A generic vector in the image of  $B$  looks like  $B\vec{v}$ . If a nonzero  $B\vec{v}$  is in  $\ker(A)$  then we know it's an eigenvector with eigenvalue 0. If a nonzero  $B\vec{v}$  is not in  $\ker(A)$ , then  $BA(B\vec{v}) = B(AB)\vec{v} = BI\vec{v} = B\vec{v}$ , and so  $B\vec{v}$  is an eigenvector with eigenvalue 1.
- (d) From (a) we know  $\text{rank}(A) = 3$ , so  $\ker(A)$  has dimension 2. Therefore 0 is an eigenvalue of  $BA$  with multiplicity 2. We also know that 1 is an eigenvalue of  $BA$  having multiplicity 3, since  $\dim(\text{Im}(B)) = \text{rank}(B) = 3$ . Therefore the characteristic polynomial of  $BA$  is:

$$f_{BA}(\lambda) = \lambda^2(\lambda - 1)^3.$$