Name: Answer Key

Math 21b Midterm 2 Tuesday, November 19th, 2002

Please circle your section.

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Question	Points	Score
1	18	
2	14	
3	12	
4	16	
5	16	
6	10	
7	14	
Total	100	

You have two hours to take this midterm. Pace yourself by keeping track of how many problems you have left to go and how much time remains. You don't have to answer the problems in any particular order. So move on to another problem if you find you're stuck and that you are spending too much time on one problem.

To receive full credit on a problem, you will need to justify your answers carefully - unsubstantiated answers, even if correct, will receive little or no credit (except if the directions for that question specifically say no justification is necessary, such as the True/False).

Please be sure to write neatly - illegible answers will also receive little or no credit.

If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of that so that the grader knows where to find your answers.

You are allowed a half page of notes on it during the test, but you are not allowed to use any other references or calculators during this test.

Good luck! Focus and do well!

Question 1. (18 points total)

True or False (3 points each) No justification is necessary, simply circle **T** or **F** for each statement.

[✓ **T**] **F** (a) If A is an invertible matrix then the kernels of A and A^{-1} must be equal.

yes, since if A is invertible, then $\ker(A) = \{\vec{0}\}$ so A^{-1} is invertible as well, $\ker(A^{-1}) = \{\vec{0}\}$ and so the kernels are equal

T [✓ **F**] (b) If A is a 5×8 matrix then it is possible for the dimension of kernel(A) to equal two.

No, a bit tricky but by rank-nullity, since rank + nullity = 8 (# of columns of A), then if nullity = 2 (if dim (kernel(A)) = 2), then rank would have to be 6, which is impossible since rank(A) $\leq$ # rows rank(A) $\leq$ # cols ... so rank ≤ 5

T [✓ **F**] (c) If a subspace V of $\mathfrak{R}^3$ contains two linearly independent vectors then V must contain at least one of the standard basis vectors as well.

No reason that it should – V might be a subspace spanned by $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\end{bmatrix}$ → a tilted plane that doesn't have any of $\vec{e}_1$, $\vec{e}_2$ or $\vec{e}_3$ in it.

T [✓ **F**] (d) If the product AB of two $n \times n$ matrices is the zero matrix, then BA must also equal the zero matrix.

Noncommutativity strikes again – for a simple counterexample consider $A = \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$ $B = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$

[✓ **T**] **F** (e) If A is a (square) orthogonal matrix, and the product AB is also orthogonal, then the matrix B must be orthogonal as well.

This is a variant on the fact that the product of two orthogonal matrices is also orthogonal. Note that if A is orthogonal, then so is $A^{-1} = A^T$ (since then $(A^{-1})^T(A^{-1}) = (A^T)^T A^{-1} = A A^{-1} = I_n$, as it should be) so A^{-1} orthogonal, AB orthogonal implies their product $A^{-1}(AB) = B$ is as well

T [✓ **F**] (f) If $A^TA = AA^T$ for an $n \times n$ matrix A, then A must be an orthogonal matrix.

No, to be orthogonal $A^TA = I_n$, but just knowing that $A^TA = AA^T$ doesn't tell us enough about A. For a counterexample, consider $A = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$, not orthogonal (why?), but $A^TA = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = AA^T$

Question 2 (14 points total)

Suppose that $A = \begin{bmatrix} 1 & 2 & 3 & -1 & 4 \\ 0 & 3 & 3 & -3 & 6 \\ 1 & 1 & 2 & 0 & 2 \\ 2 & 0 & 2 & 2 & 0 \end{bmatrix}$. Find a basis for each of the following subspaces.

(a) (6 points) Image(A). Need to find rref(A) to find pivot columns

(or simply notice linear dependence of columns ⇝ clearly cols 1, 2 are linearly independent, then notice col. 3 = col 1+2, col 4 = col 1 − 2, col 5 = 2×col 2

$$A = \left[\begin{array}{ccccc} 1 & 2 & 3 & -1 & 4 \\ 0 & 3 & 3 & -3 & 6 \\ 1 & 1 & 2 & 0 & 2 \\ 2 & 0 & 2 & 2 & 0 \end{array}\right]\begin{array}{l} \\ \div 3 \\ -(\text{I}) \\ -2(\text{I}) \end{array} \longrightarrow \left[\begin{array}{ccccc} 1 & 2 & 3 & -1 & 4 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & -1 & -1 & 1 & -2 \\ 0 & -4 & -4 & 4 & -8 \end{array}\right]\begin{array}{l} -2(\text{II}) \\ \\ +\text{II} \\ +4(\text{II}) \end{array} \longrightarrow \left[\begin{array}{ccccc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow\uparrow$ pivot cols are 1st 2 cols in matrix A

So $\text{Image}(A) = \text{span}\left(\begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}2\\3\\1\\0\end{bmatrix}\right)$ so a basis for Image(A) is $\begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}2\\3\\1\\0\end{bmatrix}$

(b) (6 points) Kernel(A)

Using rref(A):

so $x_1 + x_3 + x_4 = 0$

$x_2 + x_3 - x_4 + 2x_5 = 0$

let $x_3 = s$, $x_4 = t$, $x_5 = w$

Then $\text{kernel}(A) = \left\{ \begin{bmatrix} -s-t \\ -s+t-2w \\ s \\ t \\ w \end{bmatrix} \text{ for } s,t,w \in \mathbb{R} \right\}$

or $\text{kernel}(A) = \text{Span}\left(\begin{bmatrix}-1\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-2\\0\\0\\1\end{bmatrix}\right)$

So a basis for kernel(A) consists of the three vectors $\begin{bmatrix}-1\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-2\\0\\0\\1\end{bmatrix}$

(c) (2 points) What is the dimension of the kernel (A^T)?

We know that $\text{kernel}(A^T) = (\text{Image}(A))^\perp$, and since Image A, $(\text{Image } A)^\perp$ are both subspaces of $\mathbb{R}^4$, and $\dim(\text{Im}(A)) = 2$ from part (a) then $\dim(\text{Im}(A)^\perp) = 4-2 = 2$

so dimension of kernel (A^T) is 2.

Question 3. (12 points total)

(a) (6 points) Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a set of vectors that span a subspace V. Suppose $\vec{w}$ is another vector in the subspace V. Show that the set of vectors $\vec{w}, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ is linearly dependent.

Since the $\vec{v}_i$ span V, then since $\vec{w}$ is in V, then $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$. Now if $\vec{w} \neq \vec{0}$, then some of the constants c_i are nonzero, so $\vec{w} - c_1\vec{v}_1 - c_2\vec{v}_2 - \cdots - c_m\vec{v}_m = \vec{0}$ is a nontrivial relation among the set $\vec{w}, \vec{v}_1, \ldots, \vec{v}_m$, so it's a linearly dependent set. If $\vec{w} = \vec{0}$ then $\vec{w}$ and any other vectors is linearly dependent as $c\vec{w} + 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_m = \vec{0}$ for any nonzero constant c.

(b) (6 points) Suppose that $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ is a set of k linearly independent vectors in $\Re^n$. Suppose A is an $n \times n$ invertible matrix, and that $\vec{y}_i = A\vec{x}_i$ for i = 1 to k (i.e. $\vec{y}_1 = A\vec{x}_1, \vec{y}_2 = A\vec{x}_2, \ldots, \vec{y}_k = A\vec{x}_k$). Show that set of vectors $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_k$ is also linearly independent.

Suppose the set $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_k$ wasn't linearly independent, then $c_1\vec{y}_1 + c_2\vec{y}_2 + \cdots + c_k\vec{y}_k = \vec{0}$ for some set of constants c_i, with at least one c_i not equalling zero. But then multiplying by A^{-1} yields $A^{-1}(c_1\vec{y}_1 + \cdots + c_k\vec{y}_k = A^{-1}\vec{0}$ $= \vec{0}$ and so $c_1 A^{-1}\vec{y}_1 + \cdots + c_k A^{-1}\vec{y}_k = \vec{0}$, but since $A^{-1}\vec{y}_i = \vec{x}_i$ (since $\vec{y}_i = A\vec{x}_i$, then we get $c_1\vec{x}_1 + \cdots + c_k x_k = \vec{0}$, and since the set $\vec{x}_1, \ldots, \vec{x}_k$ is linearly independent then $c_1 = c_2 = \cdots = c_k = 0$, but then this contradicts the first assumption that the $\vec{y}_i$ were linearly dependent... so the set $\vec{y}_1, \ldots, \vec{y}_k$ must be independent as well.

Question 4. (16 points total)

(a) (10 points) Find an orthonormal basis for the kernel of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

First rref... $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$, so $x_1 = x_3 + 2x_4$
$x_2 = -2x_3 - 3x_4$

basis for kernel... $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ ... here we go! Gram-Schmidt time

$\vec{w}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, next find $\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} - \operatorname{proj}_{V_1} \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

$\nearrow$ length of $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

$V_1 = $ span of $\vec{w}_1$

$$= \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \cdot \left(\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right) \right) \left(\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{6}}\right)(8)\left(\frac{1}{\sqrt{6}}\right) \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{4}{3}\right) \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -4/3 \\ 1 \end{bmatrix}$$ now divide by its length...

length $= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{16}{9} + \frac{9}{9}} = \sqrt{30/9} = \sqrt{10/3}$,

so second unit length vector is $\frac{1}{\sqrt{10/3}} \begin{bmatrix} 2/3 \\ -1/3 \\ -4/3 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$, so an orthonormal basis for the kernel is $\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$

(b) (6 points) Suppose that $A = BC$ where B is a 4 by 3 matrix and C is a 3 by 4 matrix. Is it possible for A to be an invertible matrix? If so, give an example (write down matrices A, B and C). If not, explain why not.

No, it isn't possible. Note that $\text{kernel}(C) \neq \{\vec{0}\}$

Why? because $\text{rank}(C) \leq 3$ (less than # rows), so by rank + nullity = 4 then nullity $= \dim(\text{kernel}(C)) = 4 - \text{rank}(C) \geq 1$

Next note that anything in kernel(C) is also in kernel(BC), since if $C\vec{v} = \vec{0}$, then clearly $BC\vec{v} = \vec{0}$, so if $\vec{v} \in \text{kernel}(C)$, $\vec{v}$ is also in kernel(BC).

So $\text{kernel}(C) \neq \{\vec{0}\}$ implies $\text{kernel}(BC) \neq \{\vec{0}\}$,

so $A = BC$ is not invertible.

Question 5. (16 points total)

(a) (6 points) Write down a basis for the linear space of all skew-symmetric 3×3 matrices (recall that A is skew-symmetric if $A^T = -A$), and thus determine the dimension of this space.

Start by writing down a general 3×3 skew-symmetric matrix.

It must look like $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ (main diagonal has to be zero, as each entry equals negative itself) and a, b, c can be any numbers

So a general skew-symmetric matrix can be written as a linear combination of 3 skew-symmetric matrices: (by rewriting)

$$a\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Thus a basis consists of the three basic skew symmetric matrices $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

and so the dimension of this linear space is 3.

(b) (6 points) Find the dimension of the linear space of all symmetric $n \times n$ matrices (recall that A is symmetric if $A^T = A$)

Now write down the typical member of the space of symmetric $n\times n$ matrices: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & & \\ a_{13} & & \ddots & \\ \vdots & & & \\ a_{1n} & & & a_{nn} \end{bmatrix}$ ($a_{ij} \in \mathbb{R}$, i.e. any numbers)

Note everything below the main diagonal is just a repeat of a corresponding entry above the diagonal because of symmetricity

So a member of the $n\times n$ symmetric matrix space is determined by the choices for

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \begin{array}{l} \to n \text{ variables } a_{11}, a_{12}, \cdots a_{1n} \\ \to + n-1 \text{ variables } a_{22}, \cdots a_{2n} \\ \quad \vdots \\ + 1 \text{ variable } a_{nn} \end{array}$$

So altogether there are $n + (n-1) + \cdots + 1 = \underline{\frac{n(n+1)}{2}}$ independent variables determining a single $n\times n$ symmetric matrix $\longrightarrow$ so this equals the dimension.

(c) (4 points) If A is an $n \times n$ symmetric matrix, then is A^2 necessarily symmetric as well? Explain why or why not.

Sure... we can check without using components:

A is symmetric if and only if $A = A^T$

Since A is symmetric then $A = A^T$,

Now check to see if $(A^2)^T = A^2$, if so then A^2 is symmetric.

$$(A^2)^T = (AA)^T = \underbrace{A^TA^T = AA}_{\text{since } A = A^T} = A^2,$$

so yes A^2 is symmetric.

Question 6. (10 points total)

Use the method of least squares to find the linear function $y = mx + b$ that best fits the following data:

x	-6	-2	1	7
y	-1	2	1	6

Fitting the data for x, y to the equation $y = mx + b$ gives us four equations:

$$\begin{aligned} -1 &= m(-6) + b \\ 2 &= m(-2) + b \\ 1 &= m(1) + b \\ 6 &= m(7) + b \end{aligned} \quad \text{or} \quad \underset{A}{\begin{bmatrix} -6 & 1 \\ -2 & 1 \\ 1 & 1 \\ 7 & 1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} m \\ b \end{bmatrix}} \underset{=}{=} \underset{\vec{b}}{\begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}}$$

an inconsistent system.

normal equation $A^T A \vec{x} = A^T \vec{b}$ ...

$$A^T A = \begin{bmatrix} -6 & -2 & 1 & 7 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -6 & 1 \\ -2 & 1 \\ 1 & 1 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 90 & 0 \\ 0 & 4 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} -6 & -2 & 1 & 7 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 45 \\ 8 \end{bmatrix}$$

So now solve $\begin{bmatrix} 90 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 45 \\ 8 \end{bmatrix}$, don't even need to calculate $(A^T A)^{-1}$ as this system is so easy to solve...

... $90m = 45$ and $4b = 8$, so $m = \frac{1}{2}$, $b = 2$

and the linear function that best fits the data is $y = \frac{1}{2}x + 2$

Question 7. (14 points total)

(a) (10 points) Let T: $V \rightarrow V$ be a linear transformation, where V is a linear space. Suppose that kernel(T^2) = kernel(T). Show that it must also be the case that kernel(T^3) = kernel(T^2).

As before, a good strategy is to first show that kernel(T^2) $\subseteq$ kernel(T^3), and then show that kernel(T^3) $\subseteq$ kernel(T^2). (Note, V is a general linear space, so don't assume that V is equal to $\Re^n$ in your answer).

First show $\ker(T^2) \subseteq \ker(T^3)$. If $x \in \ker(T^2)$ then $T^2(x)=0$, but then $T^3(x) = T(T^2(x)) = T(0) = 0$, so $x \in \ker(T^3)$ as well. So $\ker(T^2) \subseteq \ker(T^3)$

Next show $\ker(T^3) \subseteq \ker(T^2)$. This is harder! Start with an element $x \in \ker(T^3)$, so $T^3(x)=0$. but $T^3(x) = T^2(T(x))$. Here's where the assumption about $\ker(T^2) = \ker(T)$ comes in... Since $T^2(T(x)) = 0$, then $T(x) \in \text{kernel}(T^2)$, but $\ker(T^2) = \text{kernel}(T)$, so if $T^2(T(x)) = 0$ then so does $T(T(x)) = 0$, so $T^2(x) = 0$. Thus if $x \in \text{kernel}(T^3)$ so $T^3(x) = 0$, then $T^2(x) = 0$, so $x \in \text{kernel}(T^2)$, so $\text{kernel}(T^3) \subseteq \text{kernel}(T^2)$.

Thus $\text{kernel}(T^2) = \text{kernel}(T^3)$

(b) (4 points) Give an example of a nonzero linear transformation T from $\Re^3$ to $\Re^3$ such that kernel(T^2) is equal to kernel(T).

Easy answer... $T(\vec{x}) = I_3 \vec{x}$, as $\text{kernel}(I_3) = \{\vec{0}\}$ and since $(I_3)^2 = I_3$ then $\ker(T^2) = \{\vec{0}\}$ as well

Harder to see... if T isn't the identity transformation then take a look at an orthogonal projection, such as onto the xy coordinate plane. kernel = z-axis, but if you do the orthogonal projection twice, the second repeat of the transformation has no effect... kernel is still z-axis. Check with matrices: $T(\vec{x}) = \underset{A}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \vec{x}$, and since $A^2 = A$, $\text{kernel}(T^2) = \ker(T)$