Math 21b Final Exam - Spring 2000

This was a three hour exam. Total points = 90.

(1) (12 pts) Let $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 & -4 \\ -5 & 6 & -7 & 8 \\ 9 & -10 & 11 & -12 \end{bmatrix}$.

(a) (2 pts) Find rref(**A**), the reduced row echelon form of **A**.
(b) (4 pts) Find bases for ker(**A**) and image(**A**).
(c) (6 pts) Find an orthonormal basis for ker(**A**).

Now let $\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}$. Show that $\mathbf{v} \in$ ker(**A**) and express **v** in terms of your orthonormal basis for ker(**A**).

(2) (10 pts) Let $\mathbf{J} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$.

We say that a 4×4 matrix **A** is symplectic if $\mathbf{AJA}^T = \mathbf{J}$.
(a) (2 pts) Show that if A is symplectic, then $\mathbf{A}^{-1}$ is symplectic.
(b) (2 pts) Show that if **A** and **B** are symplectic, then **AB** is symplectic.
(c) (2 pts) Show that **J** itself is symplectic.
(d) (4 pts) Given $\mathbf{v}, \mathbf{w} \in \mathbf{R}^4$, define $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T\mathbf{J}\mathbf{w}$. Is this an inner product on $\mathbf{R}^4$? Why or why not?

(3) (10 pts) Find all solutions to the differential equation

$$f''(t) - 2f'(t) + f(t) = 4e^{3t}.$$

Find the unique solution given the initial conditions $f(0) = 1$ and $f'(0) = 1$.

(4) (8 pts) Let $V = \{\mathbf{B} \in \mathbf{M}_n(\mathbf{R}): \mathbf{B} + \mathbf{B}^T = \mathbf{0}\}$ be a set of real $n \times n$ matrices. (Recall that such matrices are called *anti-symmetric*.) Show that V is a linear subspace of $\mathbf{M}_n(\mathbf{R})$, and find its dimension.

(5) (10 pts) Let $f \in C[-\pi, \pi]$ be given by

$$f(x) = \frac{e^x - e^{-x}}{2}.$$

(Recall that $f(x) = \sinh x$.)
Determine the Fourier coefficients of f.

(6) (14 pts) Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(a) (4 pts) Find all real/complex eigenvalues of **A** with their algebraic multiplicities.
(b) (6 pts) Does **A** have a real/complex eigenbasis? If so, find one.
(c) (2 pts) Is **A** diagonalizable? Why or why not?
(d) (2 pts) Let $T: \mathbf{R}^4 \to \mathbf{R}^4$ be the linear transformation given by $T(\mathbf{v}) = \mathbf{A}\mathbf{v}$. Describe T geometrically.

(7) (14 pts) An ecological system consists of two species whose populations at time t are given by $x(t)$ and $y(t)$. The evolution of the system is described by the equation

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(x-y+1) \\ y(y+x-3) \end{bmatrix}.$$

(a) (2 pts) Find all equilibrium points of this system in the first quadrant ($x \geq 0, y \geq 0$).
(b) (5 pts) Sketch the vector field of this system (in the first quadrant), indicating the direction of the vector field along the nullclines and inside the regions determined by the nullclines.
(c) (5 pts) Are there any stable equilibrium points? Justify your answers.
(d) (2 pts) If both species start with positive populations, can either become extinct? Explain.

(8) (12 pts) The ends of a copper wire of length π are heated so that their temperatures are $T(t, 0) = t$ and $T(t, \pi) = t + \frac{\pi^2}{2}$, respectively. The temperature of the wire at time $t = 0$ is given by $T(0, x) = \sin x + \frac{x^2}{2}$.

Assuming that $T(t, x)$ satisfies the heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$ (the special case when $\mu = 1$),

find $T(t, x)$ for all times $t \geq 0$ and all points on the wire $0 \leq x \leq \pi$.

Hints: Find a particular solution $S(t, x)$ to the heat equation (with $\mu = 1$) such that $T(t, 0) - S(t, 0) = 0$ and $T(t, \pi) - S(t, \pi) = 0$. There is such a solution $S(t, x)$ which is a polynomial in the variables t and x. Then use the linearity of solutions of the heat equation (the superposition principle) to find $T(t, x)$.