Determinants - Part II - 21b Nov. 15

Using original definition of determinant for an $n \times n$ matrix requires calculating $n!$ products (and summing them up (with +/-))

Suppose you have a computer that can do 1 billion operations per second then ...

- a 5×5 matrix requires $5! = 120$ operations which take a split second
- a 10×10 matrix requires ~ 3 million operations taking a split second
- a 15×15 ... $\sim 1 \times 10^{12}$ operations ... ~ 1000 seconds or 20 minutes
- a 20×20 ... $\sim 2 \times 10^{18}$ operations ... 2 billion seconds or ~ 77 years
- a 25×25 ... $\sim 2 \times 10^{25}$ operations ... 5 billion years
- a 30×30 ... $\sim 3 \times 10^{32}$ operations ... 8×10^{16} years! (older than our universe!)

Linearity of the determinant in rows leads to...

(a) if matrix B is obtained from matrix A by dividing a row of A by a scalar k, then

$$\det(B) = \left(\frac{1}{k}\right) \det(A)$$

(b) If B is obtained from A by adding a multiple of a row of A to another row then in fact

$$\det(B) = \det(A)$$

(c) If B is obtained from A by a row swap, then

$$\det(B) = -\det(A)$$

For (c)... row swaps... note what happens to any particular pattern while computing determinants if we swap a row...

$$\begin{bmatrix} 1 & 4 & 3 & -4 & -3 \\ 2 & 5 & 2 & -5 & -2 \\ 3 & 6 & 1 & -4 & -1 \\ 4 & 7 & 0 & -3 & 0 \\ 5 & 6 & -1 & -2 & 1 \end{bmatrix} \quad \text{swap } 1^{st} \text{ and } 2^{nd} \text{ rows} \ldots \quad \begin{bmatrix} 2 & 5 & 2 & -5 & -2 \\ 1 & 4 & 3 & -4 & -3 \\ 3 & 6 & 1 & -4 & -1 \\ 4 & 7 & 0 & -3 & 0 \\ 5 & 6 & -1 & -2 & 1 \end{bmatrix}$$

4 lines ...

3 lines → 1 less

... same thing happens if we swap any two rows...

swap 2^{nd} and 4^{th} rows...

$$\begin{bmatrix} 1 & 4 & 3 & -4 & -3 \\ 4 & 7 & 0 & -3 & 0 \\ 3 & 6 & 1 & -4 & -1 \\ 2 & 5 & 2 & -5 & -2 \\ 5 & 6 & -1 & -2 & 1 \end{bmatrix}$$

3 lines → 1 less

or swap 1^{st} and 5^{th} rows...

$$\begin{bmatrix} 5 & 6 & -1 & -2 & 1 \\ 2 & 5 & 2 & -5 & -2 \\ 3 & 6 & 1 & -4 & -1 \\ 4 & 7 & 0 & -3 & 0 \\ 1 & 4 & 3 & -4 & -3 \end{bmatrix}$$

5 lines → 1 more

... always changes by an odd number, for any pattern

<u>Example</u> Find $\det \begin{bmatrix} 0 & 3 & 2 \\ 2 & -2 & 0 \\ 1 & -1 & 2 \end{bmatrix}$ in two ways...

<u>Gauss-Jordan</u>...

$$\begin{bmatrix} 0 & 3 & 2 \\ 2 & -2 & 0 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow{\text{swap}} \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{matrix} \\ -2(\mathrm{I}) \\ \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & -4 \\ 0 & 3 & 2 \end{bmatrix} \xrightarrow{\text{swap}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{matrix} \\ \div 3 \\ \div(-4) \end{matrix}$$

$$\hookrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} +(\mathrm{II}) - 8/3(\mathrm{III}) \\ -2/3(\mathrm{III}) \\ \\ \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underline{\text{done}}$$

Tally: <u>Swaps</u> <u>Divisions</u>

→ determinant =

and check with Sarrus' Rule:

$$\begin{array}{ccccc} & & - & - & - \\ 0 & 3 & 2 & 0 & 3 \\ 2 & -2 & 0 & 2 & -2 \\ 1 & -1 & 2 & 1 & -1 \\ & & + & + & + \end{array} \rightarrow$$