Last Name:__________________________

First Name:__________________________

Mathematics 21b

Final Examination
January, 2000

Your Section (circle one):

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MWF 10	MWF 11

Question	Points	Score
1	6	
2	10	
3	10	
4	12	
5	12	
6	10	
Total	60	

No calculators are allowed.

1. [6 points] A matrix M is of the form

$$\begin{bmatrix} 0 & * & 5 & * \\ * & * & * & * \end{bmatrix},$$

where as usual the $*$'s denote unknown and possibly different real numbers. Given that M is in row-reduced echelon form, find all possible M, and explain why there are no other possibilities. For each of the M that you have found, determine its rank, image, and kernel.

2. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

a) [4 points] Construct an orthonormal eigenbasis for A.

b) [2 points] Is A diagonalizable? Why or why not?

c) [4 points] Using parts (a) and (b), compute A^{2000}.

3. Let B be the matrix

$$B = \begin{bmatrix} 1.2 & -0.4 & 0 & 0 \\ 1.3 & 0.4 & 0 & 0 \\ 0 & 0 & 1.2 & 0.4 \\ 0 & 0 & -2.6 & 0.8 \end{bmatrix}$$

a) [3 points] Find all eigenvalues of B.

b) [3 points] Does the dynamical system

$$\vec{x}(t+1) = B\vec{x}(t)$$

have a point of stable equilibrium? Why or why not?

c) [4 points] Describe qualitatively the behavior of this dynamical system if $\vec{x}(0)$ is the unit vector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

4. Consider the linear transformation $T : C^\infty \to C^\infty$ given by

$$T(f) = f'' - 2f'.$$

a) [6 points] Find all real eigenvalues of T and their corresponding eigenspaces.

b) [2 points] Let T_2 be the same linear transformation restricted to the subspace P_2 of C^∞, consisting of polynomials of degree at most 2. (That is, $T_2 : P_2 \to P_2$ is the transformation taking any polynomial f of degree at most 2 to $f'' - 2f'$.)

Choose a basis for P_2, and write the matrix A_2 of T_2 with respect to this basis.

c) [4 points] Find the image and kernel of this matrix A_2. Check (part of) your work by explaining the relationship between parts (a) and (c).

5. The following continuous dynamical system models the populations $x(t), y(t)$ of two species:

$$\frac{\partial x}{\partial t} = (y-1)x$$
$$\frac{\partial y}{\partial t} = (x+y-3)y$$

We consider only the first quadrant $x \geq 0, y \geq 0$ since we are modelling populations.

a) [2 points] Sketch the nullclines of this system, and find any equilibrium points.

b) [4 points] Use linear approximation to find the Jacobian J of the system at any equilibrium points from part (a).

c) [4 points] Analyze the stability of each equilibrium point by computing the eigenvalues of J.

d) [2 points] Sketch (approximately) the phase plane of this system, including behavior near equilibrium points and approximate direction of the flow lines in the regions separated by the nullclines.

6. Consider the temperature $T(x,t)$ of a metal bar extending from $x = 0$ to $x = \pi$. The temperature satisfies the heat equation

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$$

and the ends are held at a constant temperature of zero, i.e. $T(0,t) = 0$ and $T(\pi,t) = 0$ for all $t \geq 0$.

a) [3 points] Show that the functions $e^{-\mu n^2 t}\sin(nx)$ satisfy the differential equation and the initial conditions for all positive integers n.

b) [7 points] Suppose now that the initial temperature of the bar is given by

$$T(x,0) = \theta(x) = \sin^3 x.$$

Determine $T(x,t)$ for all $x \in [0,\pi]$ and all times $t \geq 0$. [Hint: Use Euler's formula $e^{iy} = \cos y + i \sin y$ if you are not sure about the relevant trigonometric identities.] Examine the behavior of T as $t \to \infty$.