8. The ends of a copper wire of length π are heated so that their temperatures are $T(t,0)=t$ and $T(t,\pi)=t+\frac{\pi^2}{2}$, respectively. The temperature of the wire at time $t=0$ is given by $T(0,x)=\sin x+\frac{x^2}{2}$. Assuming that $T(t,x)$ satisfies the heat equation

$$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2} \quad (\text{the special case where } \mu=1),$$

find $T(t,x)$ for all times $t\geq 0$ and all points on the wire $0\leq x\leq\pi$.

Hints: Find a particular solution $S(t,x)$ to the heat equation such that $T(t,0)-S(t,0)=0$ and $T(t,\pi)-S(t,\pi)=0$. There is such a solution $S(t,x)$ which is a polynomial in the variables t and x. Then use the linearity of solutions of the heat equation (the superposition principle) to find $T(t,x)$.

Solution. Lets find a suitable $S(t,x)$. We make the guess that $S(t,x)=\frac{x^2}{2}+f(t)$ for some f.

We want S to solve the heat equation.

$\frac{\partial S}{\partial t} = f'(t)$, $\frac{\partial^2 S}{\partial x^2} = 1$, so $f'(t) = 1$. Let's choose

$f(t) = t$, so $S(t,x) = t + \frac{x^2}{2}$.

T and S are solutions to the heat equation, so by linearity

$T - S$ is too. Define $U(t,x) = T(t,x) - S(t,x)$.

Now $U(t,0) = T(t,0) - S(t,0) = t - t = 0$.

$U(t,\pi) = T(t,\pi) - S(t,\pi) = t + \frac{\pi^2}{2} - (t + \frac{\pi^2}{2}) = 0$

$U(0,x) = T(0,x) - S(0,x) = (\sin x + \frac{x^2}{2}) - (\frac{x^2}{2}) = \sin x$.

Remember that the typical solution V to the heat equation with

boundary conditions $V(t,0) = V(t,\pi) = 0$ looks like

$V(t,x) = \sum_{n=1}^{\infty} A_n \sin nx \, e^{-n^2 t}$. We want $U(t,x)$

to be of this form; we can take $A_1 = 1$ and all other $A_n = 0$,

so $U(t,x) = \sin x \, e^{-t}$.

Now $U = T - S$, so $T = U + S$, and $\boxed{T(t,x) = e^{-t} \sin x + t + \frac{x^2}{2}.}$