FINAL EXAM **Math 21b, Fall 2003**

Name:

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
Total:		120

Problem 1) (20 points) True or False? No justifications are needed.

T	F	All symmetric real matrices are diagonalizable.
T	F	There exists a 3×3 real symmetric matrix whose Jordan-normal form is $\begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 3 \end{bmatrix}$.
T	F	If A is any matrix, then both AA^T and A^TA are orthogonally diagonalizable.
T	F	All orthogonal projections are diagonalizable.
T	F	If the regression line $y = ax + b$ obtained by fitting some data $\{(x_1, y_1), ..., (x_m, y_m)\}$ happens to contain all datapoints, then the corresponding least square solution of $A\vec{x} = \vec{b}$ is an actual solution of $A\vec{x} = \vec{b}$.
T	F	$\det(\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \end{bmatrix}) = -7$.
T	F	There exists a symmetric 2×2 matrix A such that $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.
T	F	The kernel of the operator $(D-2)^5$ is spanned by $e^{2t},\ te^{2t},\ t^2e^{2t},\ t^3e^{2t},\ t^4e^{2t}$.
T	F	Let A be a 2×2 matrix. The system $\frac{dx}{dt} = Ax$ is asymptotically stable if and only if the eigenvalues of A have negative real parts.
T	F	Let A be a 2×2 matrix. The discrete dynamical system $A(t+1) = Ax(t)$ is asymptotically stable if and only if the eigenvalues of A have negative real parts.
T	F	The subset of $X = C^\infty(\mathbf{R})$, the set of smooth functions of the real line, defined by $Y = \{f \in C^\infty(\mathbf{R}) : f(0) = 1\}$ is a linear subspace of X.
T	F	The subset of $C^\infty(\mathbf{R})$ defined by $Y = \{f \in C^\infty(\mathbf{R}) : f(0) = f''(2)\}$ is a linear subspace of X.
T	F	The operator $T(f) = (D^2 + 12tD + 17)f$ defines a linear map from $C^\infty(\mathbf{R})$ to $C^\infty(\mathbf{R})$.
T	F	The operator $T(f) = (D^2 + 12t^3D + 17t^2)f$ defines a linear map from $C^\infty(\mathbf{R})$ to $C^\infty(\mathbf{R})$.
T	F	If A is 2×2 matrix with $\det(A) < 0$, then the system $\frac{dx}{dt} = Ax$ has 0 as a stable equilibrium.
T	F	In the Fourier series expansion of the function $t+1$ on $[-\pi, \pi]$, the coefficients a_n belonging to $\cos(nt)$ are zero for all $n \geq 1$.
T	F	If a 2×2 matrix A has the eigenvalues $-2, -1$, then the orbits of system $x(t) \mapsto x(t+1) = Ax(t)$ stay bounded.
T	F	If a 2×2 matrix A has an eigenvalues $-2, -1$, then the orbits of the system $\frac{d}{dt}x(t) = Ax(t)$ stay bounded.

Problem 2) (10 points)

Match the following differential equations with the correct description. Every equation matches exactly one description. No justifications are necessary.

a)
$$\begin{aligned}\dot{x} &= 3x - 5y \\ \dot{y} &= 2x - 3y\end{aligned}$$

b)
$$\begin{aligned}\dot{x} &= -4y + 2x^2 + 2x^3 \\ \dot{y} &= 4y(1 - x^2)\end{aligned}$$

c)
$$\begin{aligned}\dot{x} &= -x + 2y - y^2 \\ \dot{y} &= 3x - y - xy - y^2\end{aligned}$$

d)
$$\begin{aligned}\dot{x} &= 3x - 5y \\ \dot{y} &= x^2 + y^2 + 2\end{aligned}$$

e)
$$\begin{aligned}\dot{x} &= 2y(x - y) - x \\ \dot{y} &= y(x - y) - y\end{aligned}$$

Fill in 1),...,5) here.

a)	b)	c)	d)	e)

1) The equation has a stable equilibrium at $x = 1, y = 1$.
2) The equation has an unstable equilibrium at $x = 1, y = 1$.
3) The equation has a non-constant solution which stays on the line $x = y$.
4) The equation has a closed periodic orbit.
5) The equation has no equilibria.

Problem 3) (10 points)

Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & -1 & -1 \end{bmatrix}$.

a) Find a basis for the kernel of A.
b) Find a basis for the image of A.

Problem 4) (10 points)

Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

a) Find all possibly complex eigenvalues of A with their algebraic multiplicities.

b) Does A have a possibly complex eigenbasis? If so, find one.
c) Is A diagonalizable? Why or why not?
d) Let T be the linear transformation defined by $T(v) = Av$. Describe T geometrically.

Problem 5) (10 points)

Find the function $f(x) = a + b\cos(x)$ which best fits the data

$$\begin{aligned} (x_1, y_1) &= (0, 1) \\ (x_2, y_2) &= (\pi/2, -1) \\ (x_3, y_3) &= (\pi, 1) \\ (x_4, y_4) &= (2\pi, 1) \end{aligned}$$

Problem 6) (10 points)

a) Find the solution of the differential equation $f'(t) + 3f(t) = e^{-2t}$, $f(0) = 0$.
b) Find the general solution of $f''(t) + 4f'(t) + 3f(t) = 1$.
with $f(0) = 1/3$, $f(1) = 1/3 + 1/e^3 - 1/e$.
c) Find the solution of $f''(t) = -4f(t)$ with $f(0) = 1$, $f'(0) = 2$.

Problem 7) (10 points)

a) (7 points) Find a 4×4 matrix A with entries $0, +1$ and -1 for which the determinant is maximal.
b) (3 points) Find the QR decomposition of A.

Problem 8) (10 points)

Define $f = \sinh(x) = \frac{e^x - e^{-x}}{2}$ on $C^\infty([-\pi, \pi])$ be a function on the interval $[-\pi, \pi]$. Find a solution $T(t, x)$ of the heat equation $\dot{T} = T_{xx}$ which satisfies $T(0, x) = f(x)$.

Hint. $\int \sinh(x)\sin(nx)\,dx = \frac{\cosh(x)\sin(nx) - n\cos(nx)\sinh(x)}{1+n^2}$. You can leave terms like $\sinh(\pi)$.

Problem 9) (10 points)

a) Find the Fourier series of $|\sin(x/2)|$ on $C([-\pi, \pi])$.

Hint. $\int \sin(x/2)\cos(nx)\,dx = \frac{-\cos((\frac{1}{2}+n)x)}{2n+1} + \frac{\cos((\frac{1}{2}-n)x)}{2n-1}$.

b) Find $\sum_{n=1}^{\infty}(-1)^n \frac{1}{4n^2-1}$.

Hint. Evaluate $f(x)$ at π.

Problem 10) (10 points)

An ecological system consists of two species whose populations at time t are given by $x(t)$ and $y(t)$. The evolution of the system is described by the equation

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(x-y+1) \\ y(x+y-3) \end{bmatrix} .$$

a) Find all equilibrium points and nullclines of this system in $x \geq 0, y \geq 0$.
b) Sketch the vector field of this system in the first quadrant $x \geq 0, y \geq 0$ indicating the direction of the vector field along the nullclines and inside the regions determined by the nullclines.
c) Are there any stable equilibrium points? Justify your answers.
d) If both species start with positive populations, can either become extinct? Explain.

Problem 11) (10 points)

Consider the linear differential equation

$$\begin{aligned} \dot{x} &= ax + y \\ \dot{y} &= ay \\ \dot{z} &= -z \,. \end{aligned}$$

a) Write the system in the form $\frac{d}{dt}\vec{x} = A\vec{x}$, where A is a matrix.
b) For which parameters a is the system stable?