COORDINATES — Math 21b, O. Knill

$\mathcal{B}$-COORDINATES. Given a basis $\vec{v}_1, \dots \vec{v}_n$, define the matrix $S = \begin{bmatrix} | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & \dots & | \end{bmatrix}$. It is invertible. If $\vec{x} = \sum_i c_i \vec{v}_i$, then c_i are called the **$\mathcal{B}$-coordinates** of $\vec{v}$. We write $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \dots \\ c_n \end{bmatrix}$. If $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$, we have $\vec{x} = S([\vec{x}]_{\mathcal{B}})$.

$\mathcal{B}$-coordinates of $\vec{x}$ are obtained by applying S^{-1} to the coordinates of the standard basis: $\boxed{[\vec{x}]_{\mathcal{B}} = S^{-1}(\vec{x})}$.

EXAMPLE. If $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, then $S = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$. A vector $\vec{v} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$ has the coordinates

$$S^{-1}\vec{v} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

Indeed, as we can check, $-3\vec{v}_1 + 3\vec{v}_2 = \vec{v}$.

EXAMPLE. Let V be the plane $x + y - z = 1$. Find a basis, in which every vector in the plane has the form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$. SOLUTION. Find a basis, such that two vectors v_1, v_2 are in the plane and such that a third vector v_3 is linearly independent to the first two. Since $(1,0,1), (0,1,1)$ are points in the plane and $(0,0,0)$ is in the plane, we can choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ which is perpendicular to the plane.

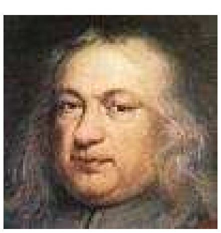
EXAMPLE. Find the coordinates of $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ with respect to the basis $\mathcal{B} = \{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$. We have $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Therefore $[v]_{\mathcal{B}} = S^{-1}\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Indeed $-1\vec{v}_1 + 3\vec{v}_2 = \vec{v}$.

$\mathcal{B}$-MATRIX. If $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis in $\mathbf{R}^n$ and T is a linear transformation on $\mathbf{R}^n$, then the $\mathcal{B}$-matrix of T is defined as

$$B = \begin{bmatrix} | & \dots & | \\ [T(\vec{v}_1)]_{\mathcal{B}} & \dots & [T(\vec{v}_n)]_{\mathcal{B}} \\ | & \dots & | \end{bmatrix}$$

COORDINATES HISTORY. Cartesian geometry was introduced by Fermat and Descartes (1596-1650) around 1636. It had a large influence on mathematics. Algebraic methods were introduced into geometry. The beginning of the vector concept came only later at the beginning of the 19'th Century with the work of Bolzano (1781-1848). The full power of coordinates becomes possible if we allow to chose our coordinate system adapted to the situation. Descartes biography shows how far dedication to the teaching of mathematics can go ...:

(...) In 1649 Queen Christina of Sweden persuaded Descartes to go to Stockholm. However the Queen wanted to draw tangents at 5 a.m. in the morning and Descartes broke the habit of his lifetime of getting up at 11 o'clock. After only a few months in the cold northern climate, walking to the palace at 5 o'clock every morning, he died of pneumonia.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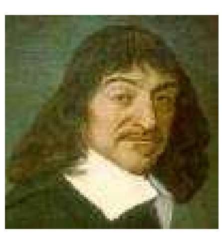
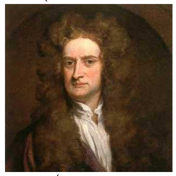
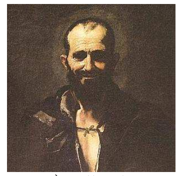

Descartes


Christina


Bolzano

CREATIVITY THROUGH LAZINESS? Legend tells that Descartes (1596-1650) introduced coordinates while lying on the bed, watching a fly (around 1630), that Archimedes (285-212 BC) discovered a method to find the volume of bodies while relaxing in the bath and that Newton (1643-1727) discovered Newton's law while lying under an apple tree. Other examples are August Kekulé's analysis of the Benzene molecular structure in a dream (a snake biting in its tail revieled the ring structure) or Steven Hawkings discovery that black holes can radiate (while shaving). While unclear which of this is actually true, there is a pattern:

According David Perkins (at Harvard school of education): "The Eureka effect", many creative breakthroughs have in common: a **long search** without apparent progress, a prevailing moment and **break through**, and finally, a transformation and **realization**. A breakthrough in a lazy moment is typical - but only after long struggle and hard work.

EXAMPLE. Let T be the reflection at the plane $x + 2y + 3z = 0$. Find the transformation matrix B in the basis $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$. Because $T(\vec{v}_1) = \vec{v}_1 = [\vec{e}_1]_{\mathcal{B}}, T(\vec{v}_2) = \vec{v}_2 = [\vec{e}_2]_{\mathcal{B}}, T(\vec{v}_3) = -\vec{v}_3 = -[\vec{e}_3]_{\mathcal{B}}$, the solution is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

SIMILARITY. The $\mathcal{B}$ matrix of A is $B = S^{-1}AS$, where $S = \begin{bmatrix} | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & \dots & | \end{bmatrix}$. One says B is **similar** to A.

EXAMPLE. If A is similar to B, then A^2+A+1 is similar to $B^2+B+\mathbf{1}$. $B = S^{-1}AS, B^2 = S^{-1}B^2S, S^{-1}S = \mathbf{1}$, $S^{-1}(A^2 + A + 1)S = B^2 + B + \mathbf{1}$.

PROPERTIES OF SIMILARITY. A, B similar and B, C similar, then A, C are similar. If A is similar to B, then B is similar to A.

QUIZZ: If A is a 2×2 matrix and let $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, What is $S^{-1}AS$?

MAIN IDEA OF CONJUGATION S. The transformation S^{-1} maps the coordinates from the standard basis into the coordinates of the new basis. In order to see what a transformation A does in the new coordinates, we map it back to the old coordinates, apply A and then map it back again to the new coordinates: $B = S^{-1}AS$.

The transformation in standard coordinates.

$$\begin{array}{ccc} \vec{v} & \stackrel{S}{\leftarrow} & \vec{w} = [\vec{v}]_{\mathcal{B}} \\ A \downarrow & & \downarrow B \\ A\vec{v} & \stackrel{S^{-1}}{\rightarrow} & B\vec{w} \end{array}$$

The transformation in $\mathcal{B}$-coordinates.

QUESTION. Can the matrix A which belongs to a projection from $\mathbf{R}^3$ to a plane $x + y + 6z = 0$ be similar to a matrix which is a rotation by 20 degrees around the z axis? No: a non-invertible A can not be similar to an invertible B: if it were, the inverse $A = SBS^{-1}$ would exist: $A^{-1} = SB^{-1}S^{-1}$.

PROBLEM. Find a clever basis for the reflection of a light ray at the line $x+2y = 0$. $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

SOLUTION. You can achieve $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ with $S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

PROBLEM. Are all shears $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ with $a \neq 0$ similar? Yes, use a basis $\vec{v}_1 = a\vec{e}_1$ and $\vec{v}_2 = \vec{e}_2$.

PROBLEM. You know $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ with $S = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. Find $e^A = 1+A+A^2+A^3/3!+\dots$. SOLUTION. Because $B^k = S^{-1}A^kS$ for every k we have $e^A = Se^BS^{-1} = \begin{bmatrix} 1/e & 0 \\ e+1/e & e \end{bmatrix}$.