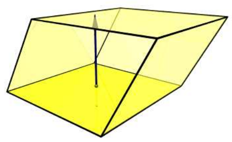
DETERMINANT AND VOLUME. If A is a $n \times n$ matrix, then $|\det(A)|$ is the volume of the n-dimensional parallelepiped E_n spanned by the n column vectors v_j of A.

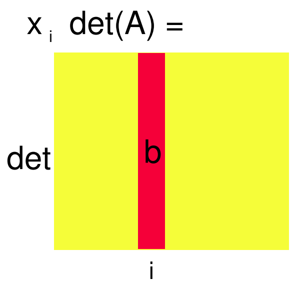


Proof. Use the QR decomposition $A = QR$, where Q is orthogonal and R is upper triangular. From $QQ^T = 1$, we get $1 = \det(Q)\det(Q^T) = \det(Q)^2$ see that $|\det(Q)| = 1$. Therefore, $\det(A) = \pm\det(R)$. The determinant of R is the product of the $||u_i|| = ||v_i - \text{proj}_{V_{j-1}} v_i||$ which was the distance from v_i to V_{j-1}. The volume $\text{vol}(E_j)$ of a j-dimensional parallelepiped E_j with base E_{j-1} in V_{j-1} and height $||u_i||$ is $\text{vol}(E_{j-1})||u_j||$. Inductively $\text{vol}(E_j) = ||u_j||\text{vol}(E_{j-1})$ and therefore $\text{vol}(E_n) = \prod_{j=1}^n ||u_j|| = \det(R)$.

The volume of a k dimensional parallelepiped defined by the vectors $v_1, \dots, v_k$ is $\sqrt{\det(A^T A)}$.

Proof. $Q^T Q = I_n$ gives $A^T A = (QR)^T(QR) = R^T Q^T Q R = R^T R$. So, $\det(R^T R) = \det(R)^2 = (\prod_{j=1}^k ||u_j||)^2$. (Note that A is a $n \times k$ matrix and that $A^T A = R^T R$ and R are $k \times k$ matrices.)

ORIENTATION. Determinants allow to **define** the orientation of n vectors in n-dimensional space. This is "handy" because there is no "right hand rule" in hyperspace... To do so, define the matrix A with column vectors v_j and define the orientation as the sign of $\det(A)$. In three dimensions, this agrees with the right hand rule: if v_1 is the thumb, v_2 is the pointing finger and v_3 is the middle finger, then their orientation is positive.



CRAMER'S RULE. This is an explicit formula for the solution of $A\vec{x} = \vec{b}$. If A_i denotes the matrix, where the column $\vec{v}_i$ of A is replaced by $\vec{b}$, then

$$x_i = \det(A_i)/\det(A)$$

Proof. $\det(A_i) = \det([v_1, \dots, b, \dots, v_n] = \det([v_1, \dots, (Ax), \dots, v_n] = \det([v_1, \dots, \sum_i x_i v_i, \dots, v_n] = x_i \det([v_1, \dots, v_i, \dots, v_n]) = x_i \det(A)$

EXAMPLE. Solve the system $5x+3y = 8, 8x+5y = 2$ using Cramers rule. This linear system with $A = \begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$. We get $x = \det \begin{bmatrix} 8 & 3 \\ 2 & 5 \end{bmatrix} = 34 y = \det \begin{bmatrix} 5 & 8 \\ 8 & 2 \end{bmatrix} = -54$.

GABRIEL CRAMER. (1704-1752), born in Geneva, Switzerland, he worked on geometry and analysis. Cramer used the rule named after him in a book "Introduction à l'analyse des lignes courbes algébraique", where he solved like this a system of equations with 5 unknowns. According to a short biography of Cramer by J.J O'Connor and E F Robertson, the rule had however been used already before by other mathematicians. Solving systems with Cramers formulas is slower than by Gaussian elimination. The rule is still important. For example, if A or b depends on a parameter t, and we want to see how x depends on the parameter t one can find explicit formulas for $(d/dt)x_i(t)$.

THE INVERSE OF A MATRIX. Because the columns of A^{-1} are solutions of $A\vec{x} = \vec{e}_i$, where $\vec{e}_j$ are basis vectors, Cramers rule together with the Laplace expansion gives the formula:

$$[A^{-1}]_{ij} = (-1)^{i+j}\det(A_{ji})/\det(A)$$

$B_{ij} = (-1)^{i+j}\det(A_{ji})$ is called the **classical adjoint** of A. **Note** the change $ij \to ji$. **Don't** confuse the classical adjoint with the **transpose** A^T which is sometimes also called the **adjoint**.

EXAMPLE. $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 4 \\ 6 & 0 & 7 \end{bmatrix}$ has $\det(A) = -17$ and we get $A^{-1} = \begin{bmatrix} 14 & -21 & 10 \\ -11 & 8 & -3 \\ -12 & 18 & -11 \end{bmatrix} /(-17)$:

$B_{11} = (-1)^{1+1}\det \begin{bmatrix} 2 & 4 \\ 0 & 7 \end{bmatrix} = 14$. $B_{12} = (-1)^{1+2}\det \begin{bmatrix} 3 & 1 \\ 0 & 7 \end{bmatrix} = -21$. $B_{13} = (-1)^{1+3}\det \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = 10$.

$B_{21} = (-1)^{2+1}\det \begin{bmatrix} 5 & 4 \\ 6 & 7 \end{bmatrix} = -11$. $B_{22} = (-1)^{2+2}\det \begin{bmatrix} 2 & 1 \\ 6 & 7 \end{bmatrix} = 8$. $B_{23} = (-1)^{2+3}\det \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} = -3$.

$B_{31} = (-1)^{3+1}\det \begin{bmatrix} 5 & 2 \\ 6 & 0 \end{bmatrix} = -12$. $B_{32} = (-1)^{3+2}\det \begin{bmatrix} 2 & 3 \\ 6 & 0 \end{bmatrix} = 18$. $B_{33} = (-1)^{3+3}\det \begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix} = -11$.

THE ART OF CALCULATING DETERMINANTS. When confronted with a matrix, it is good to go through a checklist of methods to crack the determinant. Often, there are different possibilities to solve the problem, in many cases the solution is particularly simple using one method.

- Is it a 2×2 or 3×3 matrix?
- Is it a upper or lower triangular matrix?
- Is it a partitioned matrix?
- Is it a trick: like A^{1000}?
- Does geometry imply noninvertibility?
- Do you see duplicated columns or rows?
- Can you row reduce to a triangular case?
- Are there only a few nonzero patters?
- Laplace expansion with some row or column?
- Later: Can you see the eigenvalues of $A - \lambda I_n$?

EXAMPLES.

1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 5 & 5 & 5 & 5 & 4 \\ 1 & 3 & 2 & 7 & 4 \\ 3 & 2 & 8 & 4 & 9 \end{bmatrix}$ Try row reduction.

2) $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ Laplace expansion.

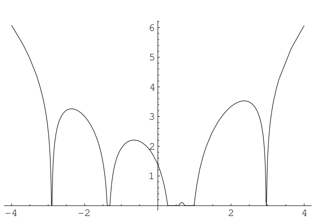
3) $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix}$ Partitioned matrix.

4) $A = \begin{bmatrix} 1 & 6 & 10 & 1 & 15 \\ 2 & 8 & 17 & 1 & 29 \\ 0 & 0 & 3 & 8 & 12 \\ 0 & 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ Make it tridiagonal.

APPLICATION HOFSTADTER BUTTERFLY. In solid state physics, one is interested in the function $f(E) = \det(L - EI_n)$, where

$$L = \begin{bmatrix} \lambda\cos(\alpha) & 1 & 0 & \cdot & 0 & 1 \\ 1 & \lambda\cos(2\alpha) & 1 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & \cdot & \cdot & 1 & \lambda\cos((n-1)\alpha) & 1 \\ 1 & 0 & \cdot & 0 & 1 & \lambda\cos(n\alpha) \end{bmatrix}$$

describes an electron in a periodic crystal, E is the energy and $\alpha = 2\pi/n$. The electron can move as a Bloch wave whenever the determinant is negative. These intervals form the **spectrum** of the quantum mechanical system. A physicist is interested in the rate of change of $f(E)$ or its dependence on λ when E is fixed. .



The graph to the left shows the function $E \mapsto \log(|\det(L - EI_n)|)$ in the case $\lambda = 2$ and $n = 5$. In the energy intervals, where this function is zero, the electron can move, otherwise the crystal is an insulator. The picture to the right shows the spectrum of the crystal depending on α. It is called the "Hofstadter butterfly" made popular in the book "Gödel, Escher Bach" by Douglas Hofstadter.