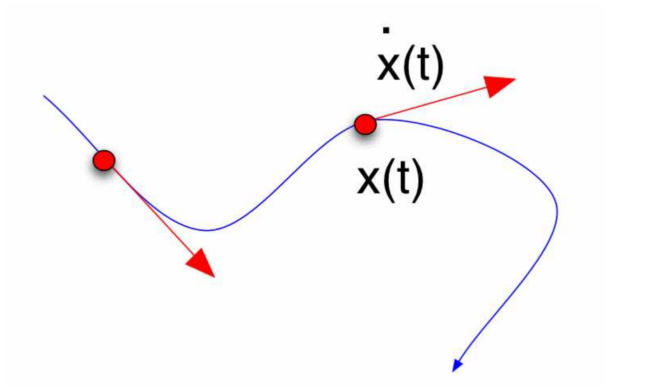
CONTINUOUS DYNAMICAL SYSTEMS I	Math 21b, O. Knill

CONTINUOUS DYNAMICAL SYSTMES. A differential equation $\frac{d}{dt}\vec{x} = f(\vec{x})$ defines a dynamical system. The solutions is a curve $\vec{x}(t)$ which has the velocity vector $f(\vec{x}(t))$ for all t. We often write $\dot{x}$ for $\frac{d}{dt}x$.

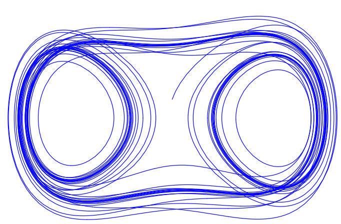


ONE DIMENSION. A system $\dot{x} = g(x,t)$ (written in the form $\dot{x} = g(x,t), \dot{t} = 1$) and often has explicit solutions

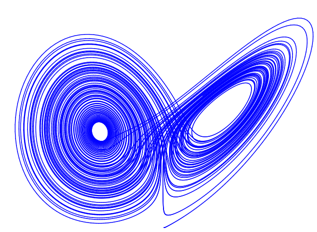
- If $\dot{x} = g(t)$, then $x(t) = \int_0^t g(t)\ dt$.
- If $\dot{x} = h(x)$, then $dx/h(x) = dt$ and so $t = \int_0^x dx/h(x) = H(x)$ so that $x(t) = H^{-1}(t)$.
- If $\dot{x} = g(t)/h(x)$, then $H(x) = \int_0^x h(x)\ dx = \int_0^t g(t)\ dt = G(t)$ so that $x(t) = H^{-1}(G(t))$.

In general, we have no closed form solutions in terms of known functions. The solution $x(t) = \int_0^t e^{-t^2}\ dt$ of $\dot{x} = e^{-t^2}$ for example can not be expressed in terms of functions $\exp, \sin, \log, \sqrt{\cdot}$ etc but it can be solved using Taylor series: because $e^{-t^2} = 1 - t^2 + t^4/2! - t^6/3! + \dots$ taking coefficient wise the anti-derivatives gives: $x(t) = t - t^3/3 + t^4/(32!) - t^7/(73!) + \dots$.

HIGHER DIMENSIONS. In higher dimensions, **chaos can set in** and the dynamical system can become unpredictable.



The nonlinear **Lorentz system** to the right $\dot{x}(t) = 10(y(t) - x(t))$, $\dot{y}(t) = -x(t)z(t) + 28x(t) - y(t)$, $\dot{z}(t) = x(t) * y(t) - 8z(t)/3$ shows a "strange attractor". Evenso completely **deterministic**: (from $x(0)$ all the path $x(t)$ is determined), there are observables which can be used as a random number generator. The **Duffing system** $\ddot{x} + \dot{x}.10 - x + x^3 - 12\cos(t) = 0$ to the left can be written in the form $\dot{v} = f(v)$ with a vector $v = (x, \dot{x}, t)$.

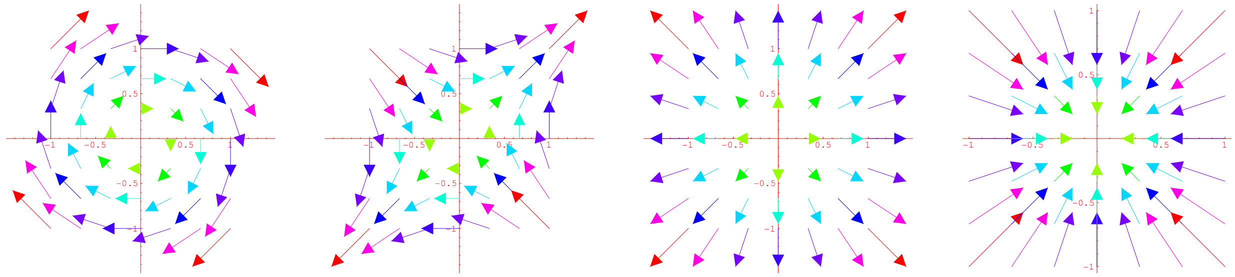


1D LINEAR DIFFERENTIAL EQUATIONS. A linear differential equation in one dimension $\boxed{\dot{x} = \lambda x}$ has the solution is $\boxed{x(t) = e^{\lambda t}x(0)}$. This differential equation appears

- as **population models** with $\lambda > 0$: birth rate of the popolation is proportional to its size.
- as a model for **radioactive decay** with $\lambda < 0$: the rate of decay is proportional to the number of atoms.

LINEAR DIFFERENTIAL EQUATIONS IN HIGHER DIMENSIONS. Linear dynamical systems have the form $\dot{x} = Ax$, where A is a matrix. Note that the origin $\vec{0}$ is an **equilibrium point**: if $x(0) = 0$, then $x(t) = 0$ for all t. The general solution is $x(t) = e^{At} = 1 + At + A^2t^2/2! + \dots$ because $\dot{x}(t) = A + 2A^2t/2! + \dots = A(1 + At + A^2t^2/2! + \dots) = Ae^{At} = Ax(t)$.
If $B = S^{-1}AS$ is diagonal with the eigenvalues $\lambda_j = a_j + ib_j$ in the diagonal, then $y = S^{-1}x$ satisfies $y(t) = e^{Bt}$ and therefore $y_j(t) = e^{\lambda_j t}y_j(0) = e^{a_j t}e^{ib_j t}y_j(0)$. The solutions in the original coordinates are $x(t) = Sy(t)$.

PHASE PORTRAITS. For differential equations $\dot{x} = f(x)$ in 2D one can **draw the vector field** $x \mapsto f(x)$. The solution $x(t)$ is tangent to the vector $f(x(t))$ everywhere. The phase portraits together with some solution curves reveal much about the system. Some examples of phase portraits of linear two-dimensional systems



UNDERSTANDING A DIFFERENTIAL EQUATION. The closed form solution like $x(t) = e^{At}x(0)$ for $\dot{x} = Ax$ is actually quite useless. One wants to understand the solution quantitatively. Questions one wants to answer are: what happens in the long term? Is the origin stable, are there periodic solutions. Can one decompose the system into simpler subsystems? We will see that **diagonalisation** allows to **understand the system:** by decomposing it into one-dimensional linear systems, which can be analyzed seperately. In general, "understanding" can mean different things:

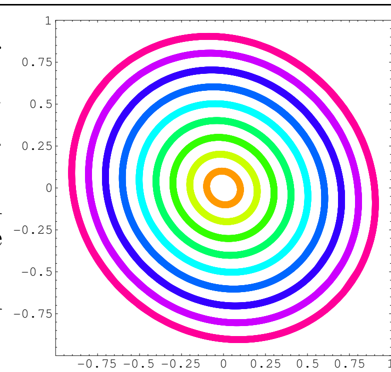
Plotting phase portraits. Computing solutions numerically and estimate the error. Finding special solutions. Predicting the shape of some orbits. Finding regions which are invariant.	Finding special closed form solutions $x(t)$. Finding a power series $x(t) = \sum_n a_n t^n$ in t. Finding quantities which are unchanged along the flow (called "Integrals"). Finding quantities which increase along the flow (called "Lyapunov functions").

LINEAR STABILITY. A linear dynamical system $\dot{x} = Ax$ with diagonalizable A is linearly stable if and only if $a_j = \mathrm{Re}(\lambda_j) < 0$ for all eigenvalues λ_j of A.

PROOF. We see that from the explicit solutions $y_j(t) = e^{a_j t}e^{ib_j t}y_j(0)$ in the basis consisting of eigenvectors. Now, $y(t) \to 0$ if and only if $a_j < 0$ for all j and $x(t) = Sy(t) \to 0$ if and only if $y(t) \to 0$.

RELATION WITH DISCRETE TIME SYSTEMS. From $\dot{x} = Ax$, we obtain $x(t+1) = Bx(t)$, with the matrix $B = e^A$. The eigenvalues of B are $\mu_j = e^{\lambda_j}$. Now $|\mu_j| < 1$ if and only if $\mathrm{Re}\lambda_j < 0$. The criterium for linear stability of discrete dynamical systems is compatible with the criterium for linear stability of $\dot{x} = Ax$.

EXAMPLE 1. The system $\dot{x} = y, \dot{y} = -x$ can in vector form $v = (x, y)$ be written as $\dot{v} = Av$, with $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The matrix A has the eigenvalues $i, -i$. After a coordinate transformation $w = S^{-1}v$ we get with $w = (a, b)$ the differential equations $\dot{a} = ia, \dot{b} = -i = b$ which has the solutions $a(t) = e^{it}a(0), b(t) = e^{-it}b(0)$. The original coordates satisfy $x(t) = \cos(t)x(0) - \sin(t)y(0)$, $y(t) = \sin(t)x(0) + \cos(t)y(0)$. Indeed e^{At} is a rotation in the plane.



EXAMPLE 2. A **harmonic oscillator** $\ddot{x} = -x$ can be written with $y = \dot{x}$ as $\dot{x} = y, \dot{y} = -x$ (see Example 1). The general solution is therefore $x(t) = \cos(t)x(0) - \sin(t)\dot{x}(0)$.

EXAMPLE 3. We take **two harmonic oscillators and couple them**: $\ddot{x}_1 = -x_1 - \epsilon(x_2 - x_1), \ddot{x}_2 = -x_2 + \epsilon(x_2 - x_1)$. For small x_i one can simulate this with two coupled penduli. The system can be written as $\ddot{v} = Av$, with $A = \begin{bmatrix} -1+\epsilon & -\epsilon \\ -\epsilon & -1+\epsilon \end{bmatrix}$. The matrix A has an eigenvalue $\lambda_1 = -1$ to the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and an eigenvalue $\lambda_2 = -1 + 2 * \epsilon$ to the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The coordinate change S is $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. It has the inverse $S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}/2$. In the coordinates $w = S^{-1}v = [y_1, y_2]$, we have oscillations $\ddot{y}_1 = -y_1$ corresponding to the case $x_1 - x_2 = 0$ (the pendula swing synchroneous) and $\ddot{y}_2 = -(1 - 2\epsilon)y_2$ corresponding to $x_1 + x_2 = 0$ (the pendula swing against each other).