COMPUTING EIGENVALUES — Math 21b, O.Knill

THE TRACE. The **trace** of a matrix A is the sum of its diagonal elements.

EXAMPLES. The trace of $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ is $1+4+8 = 13$. The trace of a skew symmetric matrix A is zero because there are zeros in the diagonal. The trace of I_n is n.

CHARACTERISTIC POLYNOMIAL. The polynomial $f_A(\lambda) = \det(\lambda I_n - A)$ is called the **characteristic polynomial** of A.

EXAMPLE. The characteristic polynomial of A above is $x^3 - 13x^2 + 15x$.

The eigenvalues of A are the roots of the characteristic polynomial $f_A(\lambda)$.

Proof. If λ is an eigenvalue of A with eigenfunction $\vec{v}$, then $A - \lambda$ has $\vec{v}$ in the kernel and $A - \lambda$ is not invertible so that $f_A(\lambda) = \det(\lambda - A) = 0$.
The polynomial has the form

$$f_A(\lambda) = \lambda^n - \mathrm{tr}(A)\lambda^{n-1} + \cdots + (-1)^n \det(A)$$

THE 2x2 CASE. The characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - (a+d)/2\lambda + (ad - bc)$. The eigenvalues are $\lambda_\pm = T/2 \pm \sqrt{(T/2)^2 - D}$, where T is the trace and D is the determinant. In order that this is real, we must have $(T/2)^2 \geq D$. Away from that parabola, there are two different eigenvalues. The map A contracts volume for $|D| < 1$.

EXAMPLE. The characteristic polynomial of $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ is $\lambda^2 - 3\lambda + 2$ which has the roots $1, 2$: $p_A(\lambda) = (\lambda - 1)(\lambda - 2)$.

THE FIBONNACCI RABBITS. The Fibonacci's recursion $u_{n+1} = u_n + u_{n-1}$ defines the growth of the rabbit population. We have seen that it can be rewritten as $\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = A \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix}$ with $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The roots of the characteristic polynomial $p_A(x) = \lambda^2 - \lambda - 1$. are $(\sqrt{5}+1)/2, (\sqrt{5}-1)/2$.

ALGEBRAIC MULTIPLICITY. If $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, where $g(\lambda_0) \neq 0$ then λ is said to be an eigenvalue of **algebraic multiplicity** k.

EXAMPLE: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ with algebraic multiplicity 2 and the eigenvalue $\lambda = 2$ with algebraic multiplicity 1.

HOW TO COMPUTE EIGENVECTORS? Because $(\lambda - A)v = 0$, the vector v is in the kernel of $\lambda - A$. We know how to compute the kernel.

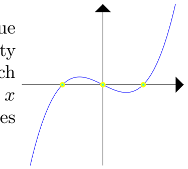
EXAMPLE FIBONNACCI. The kernel of $\lambda I_2 - A = \begin{bmatrix} \lambda_\pm - 1 & -1 \\ -1 & \lambda_\pm \end{bmatrix}$ is spanned by $\vec{v}_+ = \begin{bmatrix} (1+\sqrt{5})/2, 1 \end{bmatrix}$ and $\vec{v}_- = \begin{bmatrix} (1-\sqrt{5})/2, 1 \end{bmatrix}$. They form a basis $\mathcal{B}$.

SOLUTION OF FIBONNACCI. To obtain a formula for $A^n\vec{v}$ with $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we form $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}/\sqrt{5}$.

Now, $\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = A^n\vec{v} = A^n(\vec{v}_+/\sqrt{5} - \vec{v}_-/\sqrt{5}) = A^n\vec{v}_+/\sqrt{5} - A^n(\vec{v}_-/\sqrt{5} = \lambda_+^n\vec{v}_+/\sqrt{5} - \lambda_-^n\vec{v}_+/\sqrt{5}$. We see that $u_n = [(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]/\sqrt{5}$.

ROOTS OF POLYNOMIALS.
For polynomials of degree 3 and 4 there exist explicit formulas in terms of radicals. As Galois (1811-1832) and Abel (1802-1829) have shown, it is not possible for equations of degree 5 or higher. Still, one can compute the roots numerically.

REAL SOLUTIONS. A $(2n+1) \times (2n+1)$ matrix A always has a real eigenvalue because the characteristic polynomial $p(x) = x^5 + ... + \det(A)$ has the property that $p(x)$ goes to $\pm\infty$ for $x \to \pm\infty$. Because there exist values a, b for which $p(a) < 0$ and $p(b) > 0$, by the intermediate value theorem, there exists a real x with $p(x) = 0$. Application: A rotation in 11 dimensional space has all eigenvalues $|\lambda| = 1$. The real eigenvalue must have an eigenvalue 1 or -1.



EIGENVALUES OF TRANSPOSE. We know (see homework) that the characteristic polynomials of A and A^T agree. Therefore A and A^T have the same eigenvalues.

APPLICATION: MARKOV MATRICES. A matrix A for which each column sums up to 1 is called a **Markov matrix**. The transpose of a Markov matrix has the eigenvector $\begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix}$ with eigenvalue 1. Therefore:

A Markov matrix has an eigenvector $\vec{v}$ to the eigenvalue 1.

This vector $\vec{v}$ defines an equilibrium point of the Markov process.

EXAMPLE. If $A = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$. Then $[3/7, 4/7]$ is the equilibrium eigenvector to the eigenvalue 1.

BRETSCHERS HOMETOWN. Problem 28 in the book deals with a Markov problem in Andelfingen the hometown of Bretscher, where people shop in two shops. (Andelfingen is a beautiful village at the Thur river in the middle of a "wine country"). Initially all shop in shop W. After a new shop opens, every week 20 percent switch to the other shop M. Missing something at the new place, every week, 10 percent switch back. This leads to a Markov matrix $A = \begin{bmatrix} 8/10 & 1/10 \\ 2/10 & 9/10 \end{bmatrix}$. After some time, things will settle down and we will have certain percentage shopping in W and other percentage shopping in M. This is the equilibrium.

MARKOV PROCESS IN PROBABILITY. Assume we have a graph like a network and at each node i, the probability to go from i to j in the next step is $[A]_{ij}$, where A_{ij} is a Markov matrix. We know from the above result that there is an eigenvector $\vec{p}$ which satisfies $A\vec{p} = \vec{p}$. It can be normalized that $\sum_i p_i = 1$. The interpretation is that p_i is the probability that the walker is on the node p. For example, on a triangle, we can have the probabilities: $P(A \to B) = 1/2, P(A \to C) = 1/4, P(A \to A) = 1/4, P(B \to A) = 1/3, P(B \to B) = 1/6, P(B \to C) = 1/2, P(C \to A) = 1/2, P(C \to B) = 1/3, P(C \to C) = 1/6$. The corresponding matrix is

$$A = \begin{bmatrix} 1/4 & 1/3 & 1/2 \\ 1/2 & 1/6 & 1/3 \\ 1/4 & 1/2 & 1/6 \end{bmatrix} .$$

In this case, the eigenvector to the eigenvalue 1 is $p = [38/107, 36/107, 33/107]^T$.