FOURIER SERIES, Math 21b, O. Knill

HOW DOES IT WORK? Piecewise smooth functions $f(x)$ on $[-\pi,\pi]$ form a linear space. The numbers $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx$ are called the **Fourier coefficients** of f. We can recover f from these coefficients by $f(x) = \sum_k c_k e^{ikx}$. When we write $f = f_{even} + f_{odd}$, where $f_{even}(x) = (f(x)+f(-x))/2$ and $f_{odd}(x) = ((f(x)-f(-x))/2$, then $f_{even}(x) = a_0/\sqrt{2} + \sum_{k=1} a_k\cos(kx)$ and $f_{odd}(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$. The sum of the cos-**series** for f_{even} and the sin-**series** for f_{odd} up gives the **real Fourier series** of f. Those coefficients are $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx, a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)/\sqrt{2}\,dx$, $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$.

While the complex expansion is more elegant, the real expansion has the advantage that one does not leave the real numbers.

$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx.$
$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}.$

SUMMARY.

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)/\sqrt{2}\,dx$
$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$
$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$
$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k\sin(kx)$

EXAMPLE 1. Let $f(x) = x$ on $[-\pi,\pi]$. This is an odd function ($f(-x)+f(x)=0$) so that it has a sin series: with $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx = \frac{1}{\pi}(x\cos(kx)/k + \sin(kx)/k^2|_{-\pi}^{\pi}) = 2(-1)^{k+1}/k$ we get $x = \sum_{k=1}^{\infty} 2\frac{(-1)^{k+1}}{k}\sin(kx)$. For example, $\pi/2 = 2(1 - 1/3 + 1/5 - 1/7...)$ a **formula of Leibnitz**.
EXAMPLE 2. Let $f(x) = \cos(x) + 1/7\cos(5x)$. This **trigonometric polynomial** is already the Fourier series. The nonzero coefficients are $a_1 = 1, a_5 = 1/7$.
EXAMPLE 3. Let $f(x) = |x|$ on $[-\pi,\pi]$. This is an even function $f(-x) - f(x) = 0$ so that it has a cos series: with $a_0 = 1/(2\sqrt{2}), a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(kx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(kx)\,dx = \frac{2}{\pi}(+x\sin(kx)/k + \cos(kx)/k^2|_0^{\pi}) = \frac{2}{\pi}((-1)^k - 1)/k^2$ for $k > 0$.

WHERE ARE FOURIER SERIES USEFUL? Examples:

- **Partial differential equations**. PDE's like $\ddot{u} = c^2 u''$ become ODE's: $\ddot{u}_k = c^2k^2u_k$ can be solved as $u_k(t) = a_k\sin(ckt)u_k(0)$ and lead to solutions $u(t,x) = \sum_k a_k\sin(ckt)\sin(kx)$ of the PDE.
- **Sound** Coefficients a_k form the **frequency spectrum** of a sound f. **Filters** suppress frequencies, **equalizers** transform the Fourier space, **compressors** (i.e.MP3) select frequencies relevant to the ear.
- **Analysis:** $\sum_k a_k\sin(kx) = f(x)$ give explicit expressions for sums which would be hard to evaluate otherwise. The Leibnitz sum $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + ...$ (Example 1 above) is an example.
- **Number theory:** Example: if α is irrational, then the fact that $n\alpha$ (mod1) are uniformly distributed in $[0,1]$ can be understood with Fourier theory.
- **Chaos theory:** Quite many notions in Chaos theory can be defined or analyzed using Fourier theory. Examples are mixing properties or ergodicity.
- **Quantum dynamics:** Transport properties of materials are related to spectral questions for their Hamiltonians. The relation is given by Fourier theory.
- **Crystallography:** X ray Diffraction patterns of a crystal, analyzed using Fourier theory reveal the structure of the crystal.
- **Probability theory:** The Fourier transform $\chi_X = E[e^{iX}]$ of a random variable is called **characteristic function**. Independent case: $\chi_{x+y} = \chi_x\chi_y$.
- **Image formats:** like JPG compress by cutting irrelevant parts in Fourier space.

WHY DOES IT WORK? One has a **dot product** on functions with $f\cdot g = \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{f}(x)g(x)\,dx$. The functions $e_k = e^{ikx}$ serve as **basis vectors**. As in Euclidean space, where $v\cdot e_k = v_k$ is the **k'th coordinate** of v, the Fourier coefficient $c_k = e_k\cdot f = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}f(x)\,dx$ is the k-th coordinate of f. In the same way as we wrote $v = \sum_k v_k e_k$ in Euclidean space, we have $f(x) = \sum_k c_k e_k(x)$. The vectors e_k are orthonormal: $\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{e_k}(x)e_m(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(m-k)ix}\,dx$ which is zero for $m \neq k$ and 1 for $m = k$.

Also the functions $1/\sqrt{2}, \sin(nx), \cos(nx)$ form an **orthonormal basis** with respect to the dot product $(f,g) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$. The Fourier coefficients a_0, a_n, b_n are the coordinates of f in that basis.

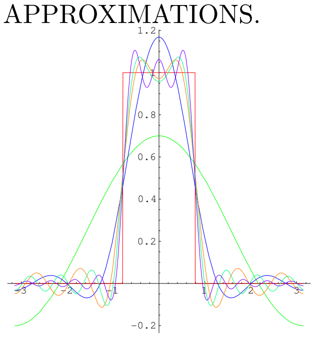
Fourier basis is not the only one. An other basis which has many applications are **Wavelet expansions**. The principle is the same.

LENGTH, DISTANCE AND ANGLE. With a dot product, one has a **length** $||f||^2 = f \cdot f = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$ and can measure **distances** between two functions as $||f-g||$ and **angles** between functions f, g by $\cos(\alpha) = f \cdot g/(||f||||g||)$. Functions are called **orthogonal** if $(f,g) = f \cdot g = 0$. For example, an even function f and an odd function g are orthogonal. The functions e^{inx} form an orthonormal family. The vectors $\sqrt{2}\cos(kx), \sqrt{2}\sin(kx)$ form an orthonormal family, $\cos(kx)$ is in the linear space of **even functions** and $\sin(kx)$ in the linear space of **odd functions**. If we work with sin or cos Fourier series, we use the dot product $\boxed{f \cdot g = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx}$ for which **the functions** $1/\sqrt{2}, \cos(kx), \sin(kx)$ **are orthonormal**.

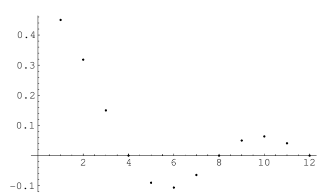
REWRITING THE DOT PRODUCT. If $f(x) = \sum_k c_k e^{ikx}$ and $g(x) = \sum_k d_k e^{ikx}$, then $\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)} g(x)\,dx = \frac{1}{2\pi}\int \sum_{k,m} \overline{c}_k e^{-ikx} d_m e^{imx} = \sum_k \overline{c_k} d_k$. The dot product is the sum of the product of the coordinates as in finite dimensions. If f and g are even, then $f = f_0/\sqrt{2} + \sum_{k=1}^{\infty} f_k \cos(kx), g = g_0/\sqrt{2} + \sum_k g_k \cos(kx)$ and $\frac{1}{\pi}\int_{\pi}^{\pi} f(x)g(x)\;dx = \sum_{k=0}^{\infty} f_k g_k$. If f and g are odd, and $f = \sum_k f_k \sin(kx)$, $g = \sum_{k=1}^{\infty} g_k \sin(kx)$ then $(f,g) = f \cdot g = \sum_{k=1}^{\infty} f_k g_k$. Especially the Perseval equality $\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 = \sum_{k=1}^{\infty} f_k^2}$ holds. It can be useful also to find closed formulas for the sum of some series.

EXAMPLE. $f(x) = x = 2(\sin(x) - \sin(2x)/2 + \sin(3x)/3 - \sin(4x)/4 + ...$ has coefficients $f_k = 2(-1)^{k+1}/k$ and so $4(1 + 1/4 + 1/9 + ...) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\;dx = 2\pi^2/3$ or $\boxed{1 + 1/4 + 1/9 + 1/16 + 1/25 + ...) = \pi^2/6}$.

APPROXIMATIONS.



If $f(x) = \sum_k b_k \cos(kx)$, then $f_n(x) = \sum_{k=1}^{n} b_k \cos(kx)$ is an approximation to f. Because $||f - f_k||^2 = \sum_{k=n+1}^{\infty} b_k^2$ goes to zero, the graphs of the functions f_n come for large n close to the graph of the function f. The picture to the left shows an approximation of a piecewise continuous even function, the right hand side the values of the coefficients $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\;dx$.



SOME HISTORY. The **Greeks** approximation of planetary motion through **epicycles** was an early use of Fourier theory: $z(t) = e^{it}$ is a circle (Aristarchus system), $z(t) = e^{it} + e^{int}$ is an epicycle (Ptolemaeus system), **18'th century** Mathematicians like Euler, Lagrange, Bernoulli knew experimentally that Fourier series worked.

Fourier's (picture left) claim of the convergence of the series was confirmed in the **19'th century** by Cauchy and Dirichlet. For continuous functions the sum does not need to converge everywhere. However, as the 19 year old Fejér (picture right) demonstrated in his theses in 1900, the coefficients still determine the function $\sum_{k=-(n-1)}^{n-1} \frac{n-|k|}{n} f_k e^{ikx} \to f(x)$ for $n \to \infty$ if f is continuous and $f(-\pi) = f(\pi)$. Partial differential equations, i.e. the **theory of heat** had sparked early research in Fourier theory.

OTHER FOURIER TRANSFORMS. On a finite interval one obtains a series, on the line an integral, on finite sets, finite sums. The **discrete Fourier transformation** (DFT) is important for applications. It can be determined efficiently by the (FFT=**Fast Fourier transform**) found in 1965, reducing the n^2 steps to $n\log(n)$.

Domain	Name	Synthesis	Coefficients
$\mathbf{T} = [-\pi, \pi)$	**Fourier series**	$f(x) = \sum_k \hat{f}_k e^{ikx}$	$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\;dx.$
$\mathbf{R} = (-\infty, \infty))$	**Fourier transforms**	$f(x) = \int_{\infty}^{\infty} \hat{f}(k)e^{ikx}\;dx$	$\hat{f}(k) = \frac{1}{2\pi}\int_{\infty}^{\infty} \hat{f}(k)e^{-ikx}\;dx$
$\mathbf{Z}_n = \{1, .., n\}$	**DFT**	$f_m = \sum_{k=1}^{n} \hat{f}_k e^{imk2\pi/n}$	$\hat{f}_k = \frac{1}{n}\sum_{m=1}^{n} f_m e^{-ikm2\pi/n}$

All these transformations can be defined in dimension d. Then $k = (k_1, ..., k_d)$ etc. are vectors. 2D DFT is for example useful in **image manipulation**.

COMPUTER ALGEBRA. Packages like Mathematica have the discrete Fourier transform built in $Fourier[\{0.3, 0.4, 0.5\}]$ for example, gives the DFT of a three vector. You can perform a simple Fourier analysis yourself by listening to a sound like Play[Sin[2000 * x * Floor[7 * x]/12], {x, 0, 20}] ...