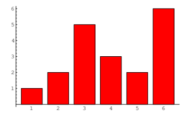
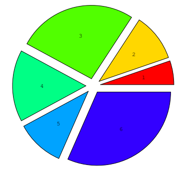
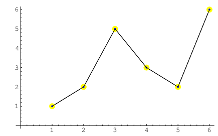
FUNCTION SPACES/LINEAR MAPS, Math 21b, O. Knill

FROM VECTORS TO FUNCTIONS. Vectors can be displayed in different ways:

$$\begin{bmatrix} 1 \\ 2 \\ 5 \\ 3 \\ 2 \\ 6 \end{bmatrix}$$



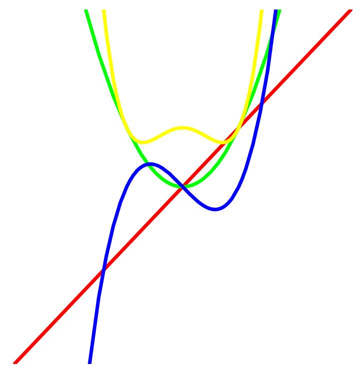




Listing the $(i, \vec{v}_i)$ can also be interpreted as the graph of a **function** $f : 1, 2, 3, 4, 5, 6 \to \mathbf{R}$, where $f(i) = \vec{v}_i$.

LINEAR SPACES. A space X in which we can add, scalar multiplications and where basic laws like commutativity, distributivity and associativity hold is called a **linear space**. Examples:

- Lines, planes and more generally, the n-dimensional Euclidean space.
- P_n, the space of all polynomials of degree n.
- The space P of all polynomials.
- C^∞, the space of all smooth functions on the line
- C^0, the space of all continuous functions on the line.
- C^1, the space of all differentiable functions on the line.
- $C^\infty(\mathbf{R}^3)$ the space of all smooth functions in space.
- L^2 the space of all functions on the line for which f^2 is integrable and $\int_\infty^\infty f^2(x)\, dx < \infty$.



In all these function spaces, the function $f(x) = 0$ which is constantly 0 is the zero function.

WHICH OF THE FOLLOWING ARE LINEAR SPACES?

- The space X of all polynomials of the form $f(x) = ax^3 + bx^4 + cx^5$
- The space X of all continuous functions on the unit interval $[-1, 1]$ which vanish at -1 and 1. It contains for example $f(x) = x^2 - |x|$.
- The space X of all smooth periodic functions $f(x+1) = f(x)$. Example $f(x) = \sin(2\pi x) + \cos(6\pi x)$.
- The space $X = \sin(x) + C^\infty(\mathbf{R})$ of all smooth functions $f(x) = \sin(x) + g$, where g is a smooth function.
- The space X of all trigonometric polynomials $f(x) = a_0 + a_1 \sin(x) + a_2 \sin(2x) + ... + a_n \sin(nx)$.
- The space X of all smooth functions on $\mathbf{R}$ which satisfy $f(1) = 1$. It contains for example $f(x) = 1 + \sin(x) + x$.
- The space X of all continuous functions on $\mathbf{R}$ which satisfy $f(2) = 0$ and $f(10) = 0$.
- The space X of all smooth functions on $\mathbf{R}$ which satisfy $\lim_{|x|\to\infty} f(x) = 0$.
- The space X of all continuous functions on $\mathbf{R}$ which satisfy $\lim_{|x|\to\infty} f(x) = 1$.
- The space X of all smooth functions on $\mathbf{R}$ of compact support: for every f, there exists an interval I such that $f(x) = 0$ outside that interval.
- The space X of all smooth functions on $\mathbf{R}^2$.

LINEAR TRANSFORMATIONS. A map T between linear spaces is called **linear** if $T(x+y) = T(x)+T(y), T(\lambda x) = \lambda T(x), T(0) = 0$. Examples:

- $Df(x)=f'(x)$ on C^∞
- $Tf(x) = \int_0^x f(x)\, dx$ on C^0
- $Tf(x) = (f(0), f(1), f(2), f(3))$ on C^∞.
- $Tf(x) = \sin(x)f(x)$ on C^∞
- $Tf(x) = (\int_0^1 f(x)g(x)\, dx)g(x)$ on $C^0[0,1]$.

WHICH OF THE FOLLOWING MAPS ARE LINEAR TRANSFORMATIONS?

- The map $T(f) = f'(x)$ on $X = C^\infty(\mathbf{T})$.
- The map $T(f) = 1 + f'(x)$ on $X = C^\infty(\mathbf{R})$.
- The map $T(f)(x) = \sin(x)f(x)$ on $X = C^\infty(\mathbf{R})$.
- The map $T(f)(x) = f(x)/x$ on $X = C^\infty(\mathbf{R})$.
- The map $T(f)(x) = \int_0^x f(x)\, dx$ on $X = C([0,1])$.
- The map $T(f)(x) = f(x+\sqrt{2})$ on $X = C^\infty(\mathbf{R})$.
- The map $T(f)(x) = \int_\infty^\infty f(x-s)\sin(s)\, ds$ on C^0.
- The map $T(f)(x) = f'(x) + f(2)$ on $C^\infty(\mathbf{R})$.
- The map $T(f)(x) = f''(x) + f(2) + 1$ on $C^\infty(\mathbf{R})$.
- The map $T(f)(x,y,z) = f_{xx}(x,y,z) + f_{yy}(x,y,z) + f_{zz}(x,y,z) + 1/(|x|)f(x,y,z)$
- The map $T(f)(x) = f(x^2)$.
- The map $T(f)(x) = f''(x) - x^2 f(x)$.
- The map $T(f)(x) = f^2(x)$ on $C^\infty(\mathbf{R})$.

EIGENVALUES, BASIS, KERNEL, IMAGE. Many concepts work also here.

X linear space	$f, g \in X, f+g \in X, \lambda f \in X, 0 \in X$.
T linear transformation	$T(f+g) = T(f) + T(g), T(\lambda f) = \lambda T(f), T(0) = 0$.
$f_1, f_2, ..., f_n$ linear independent	$\sum_i c_i f_i = 0$ implies $f_i = 0$.
$f_1, f_2, ..., f_n$ span X	Every f is of the form $\sum_i c_i f_i$.
$f_1, f_2, ..., f_n$ basis of X	linear independent and span.
T has eigenvalue λ	$Tf = \lambda f$
kernel of T	$\{Tf = 0\}$
image of T	$\{Tf \mid f \in X\}$.

Some concepts do not work without modification. Example: $\det(T)$ or $\mathrm{tr}(T)$ are not always defined for linear transformations in infinite dimensions. The concept of a basis in infinite dimensions has to be defined properly.

INNER PRODUCT. The analogue of the dot product $\sum_i f_i g_i$ for vectors $(f_1, f_2, ..., f_n), (g_1, g_2, ..., g_n)$ is the integral $\int_0^1 f(x)g(x)\, dx$ for functions on the interval $[0,1]$. One writes (f,g) and $||f|| = \sqrt{(f,f)}$. This inner product is defined on L^2 or $C^\infty([0,1])$. Example: $(x^3, x^2) = \int_0^1 x^5\, dx = 1/5$.
Having an inner product allows to define "length", "angle" in infinite dimensions. One can use it to define projections, Gram-Schmidt orthogonalization etc.