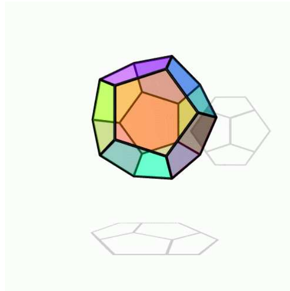
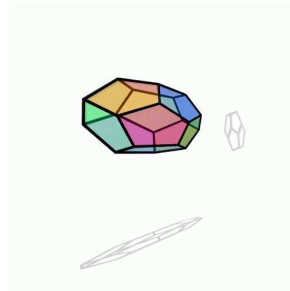
LINEAR TRANSFORMATIONS DEFORMING A BODY

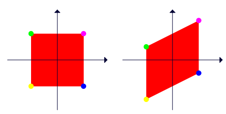




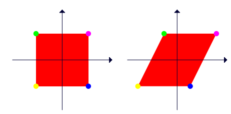
A CHARACTERIZATION OF LINEAR TRANSFORMATIONS: a transformation T from $\mathbf{R}^n$ to $\mathbf{R}^m$ which satisfies $T(\vec{0}) = \vec{0}$, $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(\lambda\vec{x}) = \lambda T(\vec{x})$ is a linear transformation.
Proof. Call $\vec{v}_i = T(\vec{e}_i)$ and define $S(\vec{x}) = A\vec{x}$. Then $S(\vec{e}_i) = T(\vec{e}_i)$. With $\vec{x} = x_1\vec{e}_1 + ... + x_n\vec{e}_n$, we have $T(\vec{x}) = T(x_1\vec{e}_1 + ... + x_n\vec{e}_n) = x_1\vec{v}_1 + ... + x_n\vec{v}_n$ as well as $S(\vec{x}) = A(x_1\vec{e}_1 + ... + x_n\vec{e}_n) = x_1\vec{v}_1 + ... + x_n\vec{v}_n$ proving $T(\vec{x}) = S(\vec{x}) = A\vec{x}$.

SHEAR:

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$



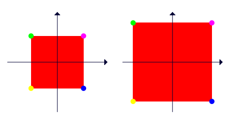
$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



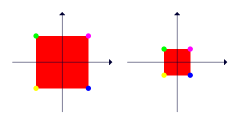
In general, shears are transformation in the plane with the property that there is a vector $\vec{w}$ such that $T(\vec{w}) = \vec{w}$ and $T(\vec{x}) - \vec{x}$ is a multiple of $\vec{w}$ for all $\vec{x}$. If $\vec{u}$ is orthogonal to $\vec{w}$, then $T(\vec{x}) = \vec{x} + (\vec{u} \cdot \vec{x})\vec{w}$.

SCALING:

$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$



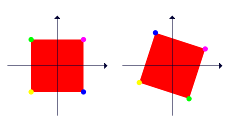
$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$



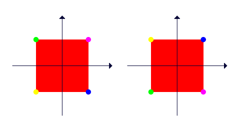
One can also look at transformations which scale x differently then y and where A is a diagonal matrix.

REFLECTION:

$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$



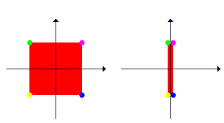
$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



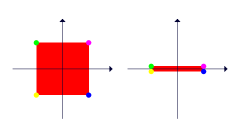
Any reflection at a line has the form of the matrix to the left. A reflection at a line containing a unit vector $\vec{u}$ is $T(\vec{x}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$ with matrix $A = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$

PROJECTION:

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$



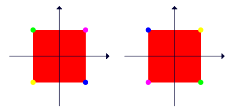
$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$



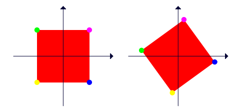
A projection onto a line containing unit vector $\vec{u}$ is $T(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$ with matrix $A = \begin{bmatrix} u_1u_1 & u_2u_1 \\ u_1u_2 & u_2u_2 \end{bmatrix}$

ROTATION:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



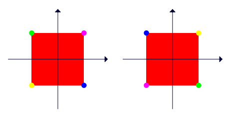
$$A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$



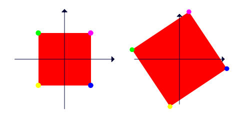
Any rotation has the form of the matrix to the right.

ROTATION-DILATION:

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$



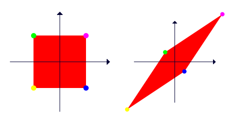
$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$



A rotation dilation is a composition of a rotation by angle $\arctan(y/x)$ and a dilation by a factor $\sqrt{x^2+y^2}$. If $z = x + iy$ and $w = a + ib$ and $T(x,y) = (X,Y)$, then $X + iY = zw$. So a rotation dilation is tied to the process of the multiplication with a complex number.

BOOST:

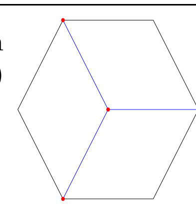
$$A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$$



The boost is a basic **Lorentz transformation** in special relativity. It acts on vectors (x, ct), where t is time, c is the speed of light and x is space.

Unlike in **Galileo transformation** $(x,t) \mapsto (x + vt, t)$ (which is a shear), time t also changes during the transformation. The transformation has the effect that it changes length (Lorentz contraction). The angle α is related to v by $\tanh(\alpha) = v/c$. One can write also $A(x, ct) = ((x + vt)/\gamma, t + (v/c^2)/\gamma x)$, with $\gamma = \sqrt{1 - v^2/c^2}$.

ROTATION IN SPACE. Rotations in space are defined by an axes of rotation and an angle. A rotation by 120° around a line containing $(0,0,0)$ and $(1,1,1)$ blongs to $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ which permutes $\vec{e}_1 \to \vec{e}_2 \to \vec{e}_3$.



REFLECTION AT PLANE. To a reflection at the xy-plane belongs the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ as can be seen by looking at the images of $\vec{e}_i$. The picture to the right shows the textbook and reflections of it at two different mirrors.

PROJECTION ONTO SPACE. To project a 4d-object into xyz-space, use for example the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The picture shows the projection of the four dimensional cube (tesseract, hypercube) with 16 edges $(\pm 1, \pm 1, \pm 1, \pm 1)$. The tesseract is the theme of the horror movie "hypercube".