IMAGE AND KERNEL — Math 21b, O. Knill

IMAGE. If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then $\{T(\vec{x}) \mid \vec{x} \in \mathbf{R}^n\}$ is called the **image** of T. If $T(\vec{x}) = A\vec{x}$, then the image of T is also called the image of A. We write $\mathrm{im}(A)$ or $\mathrm{im}(T)$.

EXAMPLES.

1) If $T(x,y,z) = (x,y,0)$, then $T(\vec{x}) = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. The image of T is the $x-y$ plane.
2) If $T(x,y)(\cos(\phi)x - \sin(\phi)y, \sin(\phi)x + \cos(\phi)y)$ is a rotation in the plane, then the image of T is the whole plane.
3) If $T(x,y,z) = x+y+z$, then the image of T is $\mathbf{R}$.

SPAN. The **span** of vectors $\vec{v}_1, \dots, \vec{v}_k$ in $\mathbf{R}^n$ is the set of all combinations $c_1\vec{v}_1 + \dots c_k\vec{v}_k$, where c_i are real numbers.

PROPERTIES.
The image of a linear transformation $\vec{x} \mapsto A\vec{x}$ is the span of the column vectors of A.
The image of a linear transformation contains 0 and is closed under addition and scalar multiplication.

KERNEL. If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then the set $\{x \mid T(x) = 0 \}$ is called the **kernel** of T. If $T(\vec{x}) = A\vec{x}$, then the kernel of T is also called the kernel of A. We write $\ker(A)$ or $\ker(T)$.

EXAMPLES. (The same examples as above)
1) The kernel is the z-axes. Every vector $(0,0,z)$ is mapped to 0.
2) The kernel consists only of the point $(0,0,0)$.
3) The kernel consists of all vector (x,y,z) for which $x+y+z=0$. The kernel is a plane.

PROPERTIES.
The kernel of a linear transformation contains 0 and is closed under addition and scalar multiplication.

IMAGE AND KERNEL OF INVERTIBLE MAPS. A linear map $\vec{x} \mapsto A\vec{x}$, $\mathbf{R}^n \mapsto \mathbf{R}^n$ is invertible if and only if $\ker(A) = \{\vec{0}\}$ if and only if $\mathrm{im}(A) = \mathbf{R}^n$.

HOW DO WE COMPUTE THE IMAGE? The rank of $\mathrm{rref}(A)$ is the dimension of the image. The column vectors of A span the image. (Dimension will be discussed later in detail).

EXAMPLES. (The same examples as above)

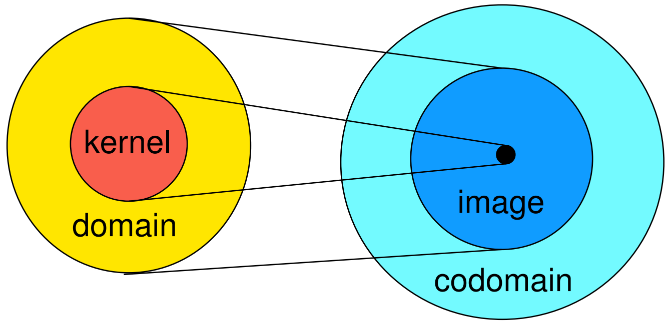
1) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ span the image.

2) $\begin{bmatrix} \cos(\phi) \\ -\sin(\phi) \end{bmatrix}$ and $\begin{bmatrix} \sin(\phi) \\ \cos(\phi) \end{bmatrix}$ span the image.

3) The 1D vector $\begin{bmatrix} 1 \end{bmatrix}$ spans the image.

HOW DO WE COMPUTE THE KERNEL? Just solve $A\vec{x} = \vec{0}$. Form $\mathrm{rref}(A)$. For every column without leading 1 we can introduce a free variable s_i. If $\vec{x}$ is the solution to $A\vec{x}_i = 0$, where all s_j are zero except $s_i = 1$, then $\vec{x} = \sum_j s_j \vec{x}_j$ is a general vector in the kernel.

EXAMPLE. Find the kernel of the linear map $\mathbf{R}^3 \to \mathbf{R}^4$, $\vec{x} \mapsto A\vec{x}$ with $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 5 \\ 3 & 9 & 1 \\ -2 & -6 & 0 \end{bmatrix}$. Gauss-Jordan elimination gives: $B = \mathrm{rref}(A) = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We see one column without leading 1 (the second one). The equation $B\vec{x} = 0$ is equivalent to the system $x + 3y = 0, z = 0$. After fixing $z = 0$, can chose $y = t$ freely and obtain from the first equation $x = -3t$. Therefore, the kernel consists of vectors $t \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. In the book, you have a detailed calculation, in a case, where the kernel is 2 dimensional.



WHY DO WE LOOK AT THE KERNEL?

- It is useful to understand linear maps. To which degree are they non-invertible?
- Helpful to understand quantitatively how many solutions a linear equation $Ax = b$ has. If x is a solution and y is in the kernel of A, then also $A(x + y) = b$, so that $x + y$ solves the system also.

WHY DO WE LOOK AT THE IMAGE?

- A solution $Ax = b$ can be solved if and only if b is in the image of A.
- Knowing about the kernel and the image is useful in the similar way that it is useful to know about the domain and range of a general map and to understand the graph of the map.

In general, the abstraction helps to understand topics like error correcing codes (Problem 53/54 in Bretschers book), where two matrices H, M with the property that $\ker(H) = \mathrm{im}(M)$ appear. The encoding $x \mapsto Mx$ is robust in the sense that adding an error e to the result $Mx \mapsto Mx + e$ can be corrected: $H(Mx + e) = He$ allows to find e and so Mx. This allows to recover $x = PMx$ with a projection P.

PROBLEM. Find $\ker(A)$ and $\mathrm{im}(A)$ for the 1×3 matrix $A = [5, 1, 4]$, a row vector.
ANSWER. $A \cdot \vec{x} = A\vec{x} = 5x + y + 4z = 0$ shows that the kernel is a plane with normal vector $[5, 1, 4]$ through the origin. The image is the codomain, which is $\mathbf{R}$.

PROBLEM. Find $\ker(A)$ and $\mathrm{im}(A)$ of the linear map $x \mapsto v \times x$, (the cross product with v.
ANSWER. The kernel consists of the line spanned by v, the image is the plane orthogonal to v.

PROBLEM. Fix a vector w in space. Find $\ker(A)$ and image $\mathrm{im}(A)$ of the linear map from $\mathbf{R}^6$ to $\mathbf{R}^3$ given by $x, y \mapsto [x, v, y] = (x \times y) \cdot w$.
ANSWER. The kernel consist of all (x, y) such that their cross product orthogonal to w. This means that the plane spanned by x, y contains w.

PROBLEM Find $\ker(T)$ and $\mathrm{im}(T)$ if T is a composition of a rotation R by 90 degrees around the z-axes with with a projection onto the x-z plane.
ANSWER. The kernel of the projection is the y axes. The x axes is rotated into the y axes and therefore the kernel of T. The image is the x-z plane.

PROBLEM. Can the kernel of a square matrix A be trivial if $A^2 = \mathbf{0}$, where $\mathbf{0}$ is the matrix containing only 0?
ANSWER. No: if the kernel were trivial, then A were invertible and A^2 were invertible and be different from $\mathbf{0}$.

PROBLEM. Is it possible that a 3×3 matrix A satisfies $\ker(A) = \mathbf{R}^3$ without $A = \mathbf{0}$?
ANSWER. No, if $A \neq \mathbf{0}$, then A contains a nonzero entry and therefore, a column vector which is nonzero.

PROBLEM. What is the kernel and image of a projection onto the plane $\Sigma : x - y + 2z = 0$?
ANSWER. The kernel consists of all vectors orthogonal to Σ, the image is the plane Σ.

PROBLEM. Given two square matrices A, B and assume $AB = BA$. You know $\ker(A)$ and $\ker(B)$. What can you say about $\ker(AB)$?
ANSWER. $\ker(A)$ is contained in $\ker(BA)$. Similar $\ker(B)$ is contained in $\ker(AB)$. Because $AB = BA$, the kernel of AB contains both $\ker(A)$ and $\ker(B)$. (It can be bigger: $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.)

PROBLEM. What is the kernel of the partitioned matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ if $\ker(A)$ and $\ker(B)$ are known?
ANSWER. The kernel consists of all vectors $(\vec{x}, \vec{y})$, where $\vec{x}$ in $\ker(A)$ and $\vec{y} \in \ker(B)$.