LINEAR TRANSFORMATIONS — Math 21b, O. Knill

TRANSFORMATIONS. A **transformation** T from a set X to a set Y is a rule, which assigns to every element in X an element $y = T(x)$ in Y. One calls X the domain and Y the codomain. A transformation is also called a **map**.

LINEAR TRANSFORMATION. A map T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is called a **linear transformation** if there is a $m \times n$ matrix A such that

$$T(\vec{x}) = A\vec{x} \; .$$

EXAMPLES.

- To the linear transformation $T(x, y) = (3x+4y, x+5y)$ belongs the matrix $\begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}$. This transformation maps the plane onto itself.
- $T(x) = -3x$ is a linear transformation from the real line onto itself. The matrix is $A = [-3]$.
- To $T(\vec{x}) = \vec{y} \cdot \vec{x}$ from $\mathbf{R}^3$ to $\mathbf{R}$ belongs the matrix $A = \vec{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$. This 1×3 matrix is also called a **row vector**. If the codomain is the real axes, one calls the map also a **function**. function defined on space.
- $T(x) = x\vec{y}$ from $\mathbf{R}$ to $\mathbf{R}^3$. $A = \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is a 3×1 matrix which is also called a **column vector**. The map defines a line in space.
- $T(x, y, z) = (x, y)$ from $\mathbf{R}^3$ to $\mathbf{R}^2$, A is the 2×3 matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. The map projects space onto a plane.
- To the map $T(x, y) = (x + y, x - y, 2x - 3y)$ belongs the 3×2 matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & -3 \end{bmatrix}$. The image of the map is a plane in three dimensional space.
- If $T(\vec{x}) = \vec{x}$, then T is called the **identity transformation**.

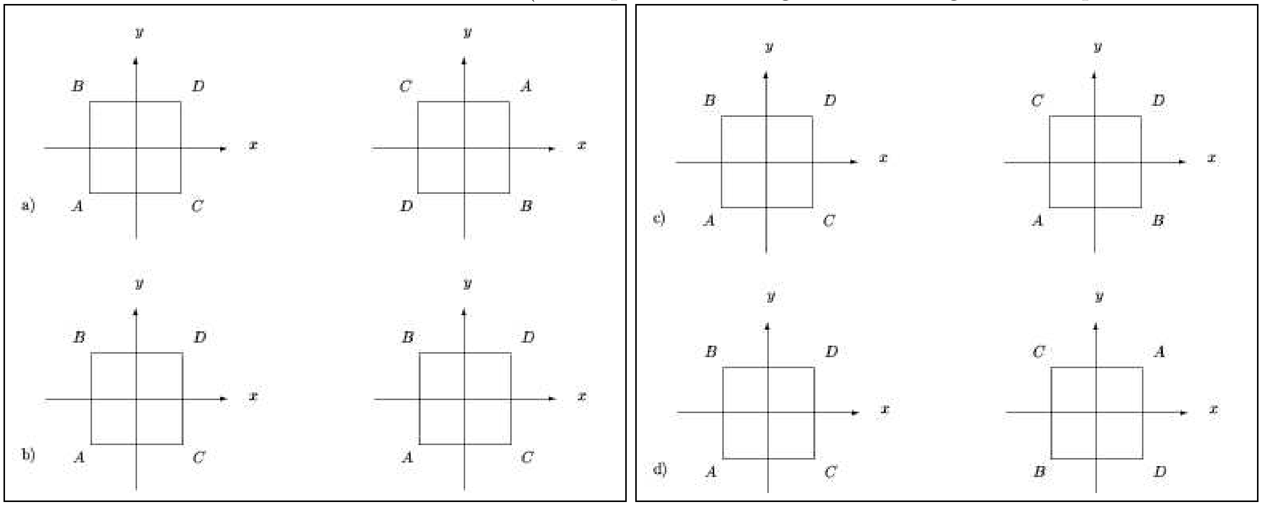
PROPERTIES OF LINEAR TRANSFORMATIONS. $T(\vec{0}) = \vec{0}$ $\quad T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ $\quad T(\lambda\vec{x}) = \lambda T(\vec{x})$

In words: Linear transformations are compatible with addition and scalar multiplication. It does not matter, whether we add two vectors before the transformation or add the transformed vectors.

ON LINEAR TRANSFORMATIONS. Linear transformations generalize the scaling transformation $x \mapsto ax$ in one dimensions.
They are important in

- geometry (i.e. rotations, dilations, projections or reflections)
- art (i.e. perspective, coordinate transformations),
- CAD applications (i.e. projections),
- physics (i.e. Lorentz transformations),
- dynamics (linearizations of general maps are linear maps),
- compression (i.e. using Fourier transform or Wavelet transform),
- coding (many codes are linear codes),
- probability (i.e. Markov processes).
- linear equations (inversion is solving the equation)

LINEAR TRANSFORMATION OR NOT? (The square to the right is the image of the square to the left):



COLUMN VECTORS. A linear transformation $T(x) = Ax$ with $A = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{bmatrix}$ has the property that the column vector $\vec{v}_1, \vec{v}_i, \vec{v}_n$ are the images of the **standard vectors** $\vec{e}_1 = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$. $\vec{e}_i = \begin{bmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ 0 \end{bmatrix}$. $\vec{e}_n = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$.

In order to find the matrix of a linear transformation, look at the image of the standard vectors and use those to build the columns of the matrix.

QUIZ.
a) Find the matrix belonging to the linear transformation, which rotates a cube around the diagonal $(1, 1, 1)$ by 120 degrees $(2\pi/3)$.
b) Find the linear transformation, which reflects a vector at the line containing the vector $(1, 1, 1)$.

INVERSE OF A TRANSFORMATION. If S is a second transformation such that $S(T\vec{x}) = \vec{x}$, for every $\vec{x}$, then S is called the **inverse** of T. We will discuss this more later.

SOLVING A LINEAR SYSTEM OF EQUATIONS. $A\vec{x} = \vec{b}$ means to invert the linear transformation $\vec{x} \mapsto A\vec{x}$. If the linear system has exactly one solution, then an inverse exists. We will write $\vec{x} = A^{-1}\vec{b}$ and see that the inverse of a linear transformation is again a linear transformation.

THE BRETSCHER CODE. Otto Bretschers book contains as a motivation a "code", where the encryption happens with the linear map $T(x, y) = (x + 3y, 2x + 5y)$. The map has the inverse $T^{-1}(x, y) = (-5x + 3y, 2x - y)$.

Cryptologists use often the following approach to crack a encryption. If one knows the input and output of some data, one often can decode the key. Assume we know, the enemy uses a Bretscher code and we know that $T(1, 1) = (3, 5)$ and $T(2, 1) = (7, 5)$. How do we get the code? The problem is to find the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

2x2 MATRIX. It is useful to decode the Bretscher code in general If $ax + by = X$ and $cx + dy = Y$, then $x = (dX - bY)/(ad - bc), y = (cX - aY)/(ad - bc)$. This is a linear transformation with matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and the corresponding matrix is $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} /(ad - bc)$.

"Switch diagonally, negate the wings and scale with a cross".