The Lorentz Boost	Math21b, Oliver Knill

This background information is not part of the course. The relation with special relativity might be fun to know about. We will use the functions $\cosh(x) = (e^x + e^{-x})/2, \sinh(x) = (e^x - e^- x)/2$ on this page.

LORENTZ BOOST. The linear transformation of the plane given by the matrix

$$A = \begin{bmatrix} \cosh(\phi) & \sinh(\phi) \\ \sinh(\phi) & \cosh(\phi) \end{bmatrix}$$

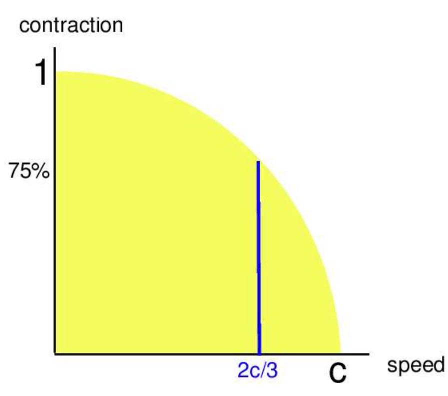
is called the **Lorentz boost**. The transformation $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix}$ with $y = ct$ appears in physics.

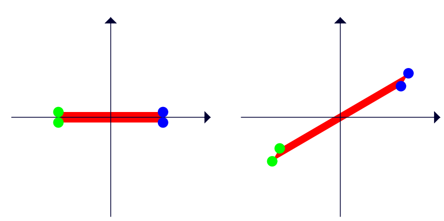
PHYSICAL INTERPRETATIONS. In classical mechanics, when a particle travels with velocity v on the line, its new position satisfies $\tilde{x} = x + tv$.
According to special relativity, this is only an approximation. In reality, the motion is described by $\begin{pmatrix} x \\ ct \end{pmatrix} \mapsto A \begin{pmatrix} x \\ ct \end{pmatrix}$ where A is the above matrix and where the angle ϕ is related to v by the formula $\tanh(\phi) = v/c$. Trigonometric identities give $\sinh(\phi) = (v/c)/\gamma, \cosh(\phi) = 1/\gamma$, where $\gamma = \sqrt{1 - v^2/c^2}$. The linear transformation tells then that $A(x, ct) = ((x + vt)/\gamma, t + (v/c^2)/\gamma x)$. For small velocities v, the value of γ is close to 1 and v/c^2 is close to zero so that $A(x, ct)$ is close to $(x + vt, t)$.

LORENTZ CONTRACTION. If we displace a ruler $[a, b]$ with velocity v then its end points are not $[a + tv, b + tv]$ as Newtionian mechanics would tell but $[(a + tv)/\gamma, (b + tv)/\gamma]$. The ruler is by a factor $1/\gamma$ larger, when we see it in a coordinate system where it rests. The constant γ is called the **Lorentz contraction** factor.

For example, for $v = 2/3c$, where c is the speed of light, the contraction is 75 percent. The following picture shows the ruler in the moving coordinate system and in the resting coordinate system.

In the resing coordinate system, the two end points of the ruler have a different time. If a light signal would be sent out simultaneously at the both ends, then this signal would reach the origin at different times. The one to the left earlier than the one to the right. The end point to the left is "younger".)





MAGNETIC FORCE DERIVED FROM ELECTRIC FORCE.
One striking application of the Lorentz transformation is that if you take two wires and let an electric current flow in the same direction, then the distance between the electrons shrinks: the positively charged ions in the wire see a larger electron density than the ion density. The two wires appear negatively "charged" and repel each other. If the currents flow in different directions and we go into a coordinate system, where the electrons are at rest in the first wire, then the ion density of the ions in the same wire appears denser as well as the electron density in the other wire. The two wires then attract each other. The force is proportional to $1/r$.