NONLINEAR DYNAMICAL SYSTEMS	Math 21b, O. Knill

SUMMARY. For linear systems $\dot{x} = Ax$, the eigenvalues of A determine the behavior completely. For nonlinear systems explicit formulas for solutions are no more available in general. It even also happen that orbits go go off to infinity in finite time like in $\dot{x} = x^2$ with solution $x(t) = -1/(t - x(0))$. With $x(0) = 1$ it reaches infinity at time $t = 1$. Linearity is often too crude. The exponential growth $\dot{x} = ax$ of a bacteria colony for example is slowed down due to lack of food and the **logistic model** $\dot{x} = ax(1 - x/M)$ would be more accurate, where M is the population size for which bacteria starve so much that the growth has stopped: $x(t) = M$, then $\dot{x}(t) = 0$. Nonlinear systems can be investigated with **qualitative methods**. In 2 dimensions $\dot{x} = f(x, y), \dot{y} = g(x, y)$, where chaos does not happen, the analysis of **equilibrium points** and **linear approximation** at those points in general allows to understand the system. Also in higher dimensions, where ODE's can have chaotic solutions, the analysis of equilibrium points and linear approximation at those points is a place, where linear algebra becomes useful.

EQUILIBRIUM POINTS. A point x_0 is called an **equilibrium point** of $\dot{x} = f(x)$ if $f(x_0) = 0$. If $x(0) = x_0$ then $x(t) = x_0$ for all times. The system $\dot{x} = x(6-2x-y), \dot{y} = y(4-x-y)$ for example has the four equilibrium points $(0, 0), (3, 0), (0, 4), (2, 2)$.

JACOBIAN MATRIX. If x_0 is an equilibrium point for $\dot{x} = f(x)$ then $[A]_{ij} = \frac{\partial}{\partial x_j} f_i(x)$ is called the **Jacobian** at x_0. For two dimensional systems

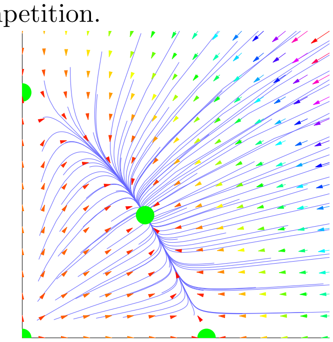
$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \qquad \text{this is the } 2 \times 2 \text{ matrix} \qquad A = \begin{bmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{bmatrix}.$$

The linear ODE $\dot{y} = Ay$ with $y = x - x_0$ approximates the nonlinear system well near the equilibrium point. The Jacobian is the linear approximation of $F = (f, g)$ near x_0.

VECTOR FIELD. In two dimensions, we can draw the vector field by hand: attaching a vector $(f(x, y), g(x, y))$ at each point (x, y). To find the equilibrium points, it helps to draw the **nullclines** $\{f(x, y) = 0\}, \{g(x, y) = 0\}$. The equilibrium points are located on intersections of nullclines. The eigenvalues of the Jacobeans at equilibrium points allow to draw the vector field near equilibrium points. This information is sometimes enough to draw the vector field **by hand**.

MURRAY SYSTEM (see handout) $\dot{x} = x(6 - 2x - y), \dot{y} = y(4 - x - y)$ has the nullclines $x = 0, y = 0, 2x + y = 6, x + y = 5$. There are 4 equilibrium points $(0, 0), (3, 0), (0, 4), (2, 2)$. The Jacobian matrix of the system at the point (x_0, y_0) is $\begin{bmatrix} 6 - 4x_0 - y_0 & -x_0 \\ -y_0 & 4 - x_0 - 2y_0 \end{bmatrix}$. Note that without interaction, the two systems would be logistic systems $\dot{x} = x(6 - 2x), \dot{y} = y(4 - y)$. The additional $-xy$ is the competition.

Equilibrium	Jacobean	Eigenvalues	Nature of equilibrium
(0,0)	$\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$	$\lambda_1 = 6, \lambda_2 = 4$	Unstable source
(3,0)	$\begin{bmatrix} -6 & -3 \\ 0 & 1 \end{bmatrix}$	$\lambda_1 = -6, \lambda_2 = 1$	Hyperbolic saddle
(0,4)	$\begin{bmatrix} 2 & 0 \\ -4 & -4 \end{bmatrix}$	$\lambda_1 = 2, \lambda_2 = -4$	Hyperbolic saddle
(2,2)	$\begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix}$	$\lambda_i = -3 \pm \sqrt{5}$	Stable sink



USING TECHNOLOGY (Example: Mathematica). Plot the vector field:

```
Needs["Graphics`PlotField`"]
f[x_,y_]:={x(6-2x-y),y(5-x-y)};PlotVectorField[f[x,y],{x,0,4},{y,0,4}]
```

Find the equilibrium solutions:

```
Solve[{x(6-2x-y)==0,y(5-x-y)==0},{x,y}]
```

Find the Jacobian and its eigenvalues at $(2, 2)$:

```
A[{x_,y_}]:={{6-4x,-x},{-y,5-x-2y}};Eigenvalues[A[{2,2}]]
```

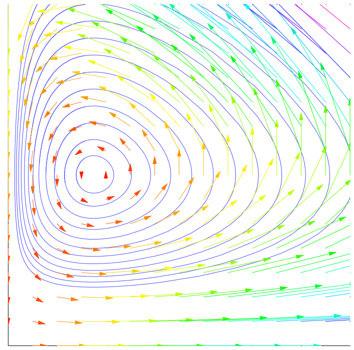
Plotting an orbit:

```
S[u_,v_]:=NDSolve[{x'[t]==x[t](6-2x[t]-y[t]),y'[t]==y[t](5-x[t]-y[t]),x[0]==u,y[0]==v},{x,y},{t,0,1}]
ParametricPlot[Evaluate[{x[t],y[t]}/.S[0.3,0.5]],{t,0,1},AspectRatio->1,AxesLabel->{"x[t]","y[t]"}]
```

VOLTERRA-LODKA SYSTEMS are systems of the form

$$\begin{aligned}\dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy\end{aligned}$$

This example has equilibrium points $(0,0)$ and $(1/2,1)$.



It describes for example a tuna shark population. The tuna population $x(t)$ becomes smaller with more sharks. The shark population grows with more tuna. Volterra explained so first the oscillation of fish populations in the Mediterrian sea.

EXAMPLE: HAMILTONIAN SYSTEMS are systems of the form

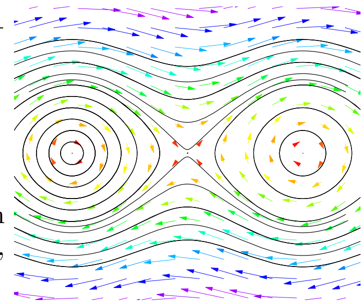
$$\begin{aligned}\dot{x} &= \partial_y H(x,y) \\ \dot{y} &= -\partial_x H(x,y)\end{aligned}$$

where H is called the **energy**. Usually, x is the position and y the momentum.

THE PENDULUM: $H(x,y) = y^2/2 - \cos(x)$.

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\sin(x)\end{aligned}$$

x is the angle between the pendulum and y-axes, y is the angular velocity, $\sin(x)$ is the potential.



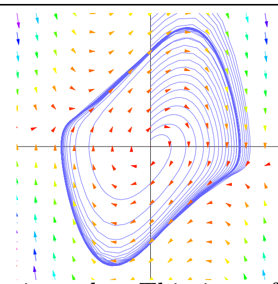
(See homework). Hamiltonian systems preserve energy $H(x,y)$ because $\frac{d}{dt}H(x(t),y(t)) = \partial_x H(x,y)\dot{x} + \partial_y H(x,y)\dot{y} = \partial_x H(x,y)\partial_y H(x,y) - \partial_y H(x,y)\partial_x H(x,y) = 0$. Orbits stay on level curves of H.

EXAMPLE: LIENHARD SYSTEMS are differential equations of the form $\ddot{x} + \dot{x}F'(x) + G'(x) = 0$. With $y = \dot{x} + F(x), G'(x) = g(x)$, this gives

$$\begin{aligned}\dot{x} &= y - F(x) \\ \dot{y} &= -g(x)\end{aligned}$$

VAN DER POL EQUATION $\ddot{x} + (x^2 - 1)\dot{x} + x = 0$ appears in electrical engineering, biology or biochemistry. Since $F(x) = x^3/3 - x, g(x) = x$.

$$\begin{aligned}\dot{x} &= y - (x^3/3 - x) \\ \dot{y} &= -x\end{aligned}$$

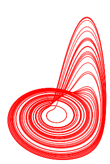


Lienhard systems have **limit cycles**. A trajectory always ends up on that limit cycle. This is useful for engineers, who need oscillators which are stable under changes of parameters. One knows: if $g(x) > 0$ for $x > 0$ and F has exactly three zeros $0, a, -a$, $F'(0) < 0$ and $F'(x) \geq 0$ for $x > a$ and $F(x) \to \infty$ for $x \to \infty$, then the corresponding Lienhard system has exactly one stable limit cycle.

CHAOS can occur for systems $\dot{x} = f(x)$ in three dimensions. For example, $\ddot{x} = f(x,t)$ can be written with $(x,y,z) = (x,\dot{x},t)$ as $(\dot{x},\dot{y},\dot{z}) = (y, f(x,z), 1)$. The system $\ddot{x} = f(x,\dot{x})$ becomes in the coordinates $(x,\dot{x})$ the ODE $\dot{x} = f(x)$ in four dimensions. The term **chaos** has no uniform definition, but usually means that one can find a copy of a random number generator embedded inside the system. Chaos theory is more than 100 years old. Basic insight had been obtained by Poincaré. During the last 30 years, the subject exploded to its own branch of physics, partly due to the availability of computers.

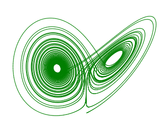
ROESSLER SYSTEM

$$\begin{aligned}\dot{x} &= -(y+z) \\ \dot{y} &= x + y/5 \\ \dot{z} &= 1/5 + xz - 5.7z\end{aligned}$$



LORENTZ SYSTEM

$$\begin{aligned}\dot{x} &= 10(y-x) \\ \dot{y} &= -xz + 28x - y \\ \dot{z} &= xy - \frac{8z}{3}\end{aligned}$$

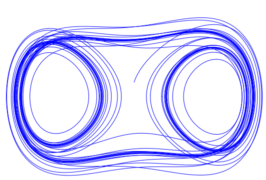


These two systems are examples, where one can observe **strange attractors**.

THE DUFFING SYSTEM

$\ddot{x} + \frac{\dot{x}}{10} - x + x^3 - 12\cos(t) = 0$

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -y/10 - x + x^3 - 12\cos(z) \\ \dot{z} &= 1\end{aligned}$$



The Duffing system models a metallic plate between magnets. Other chaotic examples can be obtained from mechanics like the **driven pendulum** $\ddot{x} + \sin(x) - \cos(t) = 0$.