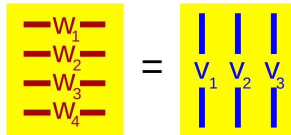
ON SOLUTIONS OF LINEAR EQUATIONS — Math 21b, O. Knill

Homework Section 1.3: 4,14,34,48,50,26*,46* , due 9/29/2003

MATRIX. A rectangular array of numbers is called a **matrix**.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

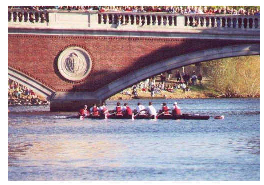


A matrix with m **rows** and n **columns** is called a $m \times n$ matrix. A matrix with one column is a **column vector**. The entries of a matrix are denoted a_{ij}, where i is the row number and j is the column number.

ROW AND COLUMN PICTURE. Two interpretations

$$A\vec{x} = \begin{bmatrix} -\vec{w}_1- \\ -\vec{w}_2- \\ \cdots \\ -\vec{w}_m- \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \cdots \\ \vec{w}_m \cdot \vec{x} \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{b} .$$



"Row and Column at Harvard"

Row picture: each b_i is the dot product of a row vector $\vec{w}_i$ with $\vec{x}$.
Column picture: $\vec{b}$ is a sum of scaled column vectors $\vec{v}_j$.

EXAMPLE. The system of linear equations

$$\left| \begin{array}{ll} 3x - 4y - 5z & = 0 \\ -x + 2y - z & = 0 \\ -x - y + 3z & = 9 \end{array} \right|$$

is equivalent to $A\vec{x} = \vec{b}$, where A is a **coefficient matrix** and $\vec{x}$ and $\vec{b}$ are **vectors**.

$$A = \begin{bmatrix} 3 & -4 & -5 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} .$$

The **augmented matrix** (separators for clarity)

$$B = \begin{bmatrix} 3 & -4 & -5 & | & 0 \\ -1 & 2 & -1 & | & 0 \\ -1 & -1 & 3 & | & 9 \end{bmatrix} .$$

In this case, the row vectors of A are
$\vec{w}_1 = \begin{bmatrix} 3 & -4 & -5 \end{bmatrix}$
$\vec{w}_2 = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$
$\vec{w}_3 = \begin{bmatrix} -1 & -1 & 3 \end{bmatrix}$
The column vectors are

$$\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -4 \\ -2 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$$

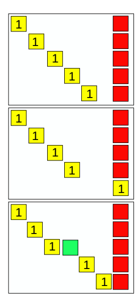
Row picture:

$$0 = b_1 = \begin{bmatrix} 3 & -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Column picture:

$$\begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

SOLUTIONS OF LINEAR EQUATIONS. A system $A\vec{x} = \vec{b}$ with m equations and n unknowns is defined by the $m \times n$ matrix A and the vector $\vec{b}$. The row reduced matrix rref(B) of B determines the number of solutions of the system $Ax = b$. There are three possibilities:

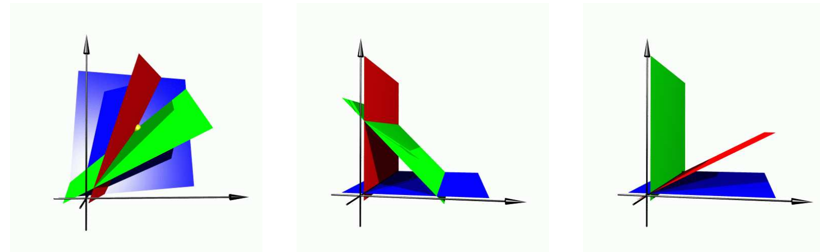


- **Consistent: Exactly one solution.** There is a leading 1 in each column of A but none in the last column of the augmented matrix B.
- **Inconsistent: No solutions.** There is a leading 1 in the last row of B.
- **Consistent: Infinitely many solutions.** There are columns in A without leading 1.

If $m < n$ (less equations then unknowns), then there are either zero or infinitely many solutions.

The **rank** rank(A) of a matrix A is the number of leading ones in rref(A).

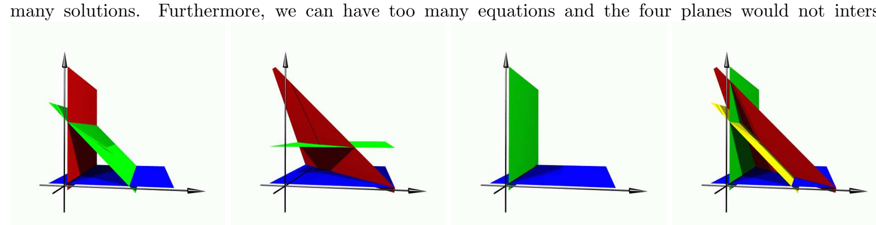
(exactly one solution) (no solution) (infinitely many solutions)



MURPHYS LAW.

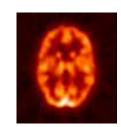
"If anything can go wrong, it will go wrong".
"If you are feeling good, don't worry, you will get over it!"
"Whenever you do Gauss-Jordan elimination, you screw up during the first couple of steps."

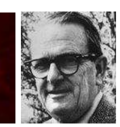
MURPHYS LAW IS TRUE. Two equations could contradict each other. Geometrically this means that the two planes do not intersect. This is possible if they are parallel. Even without two planes being parallel, it is possible that there is no intersection between all three of them. Also possible is that we don't have enough equations (for example because two equations are the same) and that there are many solutions. Furthermore, we can have too many equations and the four planes would not intersect.



RELEVANCE OF EXCEPTIONAL CASES. There are important applications, where "unusual" situations happen: For example in medical tomography, systems of equations appear which are "ill posed". In this case one has to be careful with the method.

The linear equations are then obtained from a method called the **Radon transform**. The task for finding a good method had led to a Nobel prize in Medicis 1979 for Allan Cormack. Cormack had sabbaticals at Harvard and probably has done part of his work on tomography here. Tomography helps today for example for cancer treatment.





MATRIX ALGEBRA. Matrices can be added, subtracted if they have the same size:

$$A+B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}+b_{m2} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

They can also be scaled by a scalar λ:

$$\lambda A = \lambda \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix}$$