EIGENVALUES & DYNAMICAL SYSTEMS **21b,O.Knill**

HOMEWORK: Section 7.1: 38,50, Section 7.2: 8,28,38,25*,26*

EIGENVALUES AND EIGENVECTORS. A nonzero vector v is called an **eigenvector** of A with **eigenvalue** λ if $\boxed{Av = \lambda v}$.

EXAMPLES.

- $\vec{v}$ is an eigenvector to the eigenvalue 0 if $\vec{v}$ is in the kernel of A.
- A rotation in space has an eigenvalue 1 (homework).
- If A is a diagonal matrix with diagonal elements a_i, $\vec{e}_i$ is an eigenvector with eigenvalue a_i.
- A shear A in the direction v has an eigenvector $\vec{v}$.
- Projections have eigenvalues 1 or 0.
- Reflections have eigenvalues 1 or -1.
- A rotation in the plane by an angle 30 degrees has no eigenvector. (The actual eigenvectors are complex).

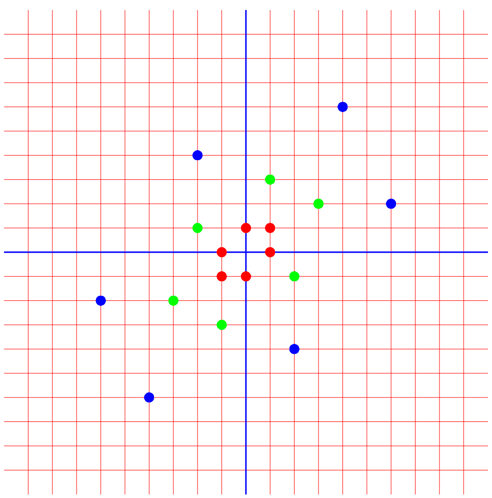
LINEAR DYNAMICAL SYSTEMS.
Iterating a linear map $x \mapsto Ax$ is called a **discrete dynamical system**. One wants to understand what happens with $x_1 = Ax, x_2 = AAx = A^2x, x_3 = AAAx = A^3x, \dots$.

EXAMPLE 1: $x \mapsto ax$ or $x_{n+1} = ax_n$ has the solution $x_n = a^n x_0$. For example, $1.03^{20} \cdot 1000 = 1806.11$ is the balance on a bank account which had 1000 dollars 20 years ago and if the interest rate was constant 3 percent.
EXAMPLE 2: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $A\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A^2\vec{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. $A^3\vec{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. $A^4\vec{v} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ etc.
EXAMPLE 3: If $\vec{v}$ is an eigenvector with eigenvalue λ, then $A\vec{v} = \lambda\vec{v}, A^2\vec{v} = A(A\vec{v})) = A\lambda\vec{v} = \lambda A\vec{v} = \lambda^2\vec{v}$ and more generally $A^n\vec{v} = \lambda^n\vec{v}$.

RECURSION: If a scalar quantity u_{n+1} does not only depend on u_n but also on u_{n-1} we can write $(x_n, y_n) = (u_n, u_{n-1})$ and get a linear map because x_{n+1}, y_{n+1} depend in a linear way on x_n, y_n.

A RECURSION PROBLEM. A linear recursion problem which appears in quantum mechanics is $u_{n+1} + u_{n-1} = Eu_n$ and $u_0 = 0, u_1 = 1$. Because $\begin{bmatrix} E & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix}$. The recursion is done by iterating the matrix A. Lets take $E = 1$:
$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$
$A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. We see that A^3 is a reflection at the origin which has the eigenvalue $\lambda = -1$ and A^6 is the identity. Every initial vector is mapped after 6 iterations back to its original starting point.

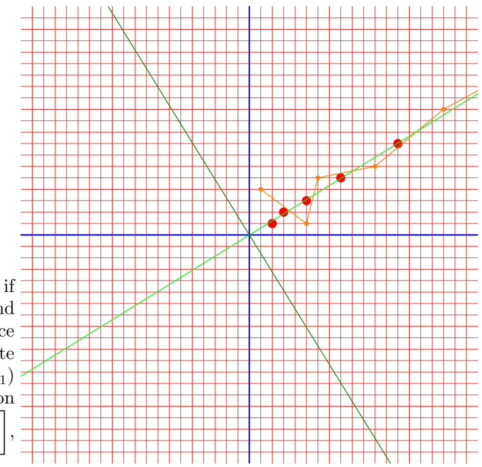
If the E parameter is changed, the dynamics also changes. For $E = 3$ for example, most initial points will escape to infinity similar as in the next example. Indeed, for $E = 3$, there is an eigenvector $\vec{v} = (3 + \sqrt{5})/2$ to the eigenvalue $\lambda = (3 + \sqrt{5})/2$ and $A^n\vec{v} = \lambda^n\vec{v}$ escapes to ∞.



THE FIBONNACCI RECURSION:
In the third section of Liber abbaci, published in 1202 **Leonardo Fibonacci** (1170-1250) writes:

> *A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?*

Mathematically, how does u_n grow, if $u_{n+1} = u_n + u_{n-1}$? We can assume $u_0 = 1$ and $u_1 = 2$ to match Leonardos example. The sequence is $(1, 2, 3, 5, 8, 13, 21, \dots)$. As before we can write this recursion using vectors $(x_n, y_n) = (u_n, u_{n-1})$ starting with $(1, 2)$. The matrix A to this recursion is $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Iterating gives $A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $A^2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = A\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.



SOLUTION KNOWING EIGENSYSTEM. If $A\vec{v}_1 = \lambda_1\vec{v}_1, A\vec{v}_2 = \lambda_2\vec{v}_2$ and $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$, we have an explicit solution $A^n\vec{v} = c_1\lambda_1^n\vec{v}_1 + c_2\lambda_2^n\vec{v}_2$. This motivates to find good methods to compute eigenvalues and eigenvectors.

EVOLUTION OF QUANTITIES. Example could be market systems, population quantities of different species, or ingredient quantities in a chemical reaction. A linear description might not always be be a good model but it has the advantage that we can solve the system explicitly. Eigenvectors will provide the key to do so. You do a biological problem like this in the homework.

EXAMPLE 1: Quantum mechanics. Some quantum mechanical systems of a particle in a potential V are described by $(Lu)_n = u_{n+1} + u_{n-1} + V_n u_n$. Energies E for which $(Lu)_n = Eu_n$, we have the recursion $u_{n+1} + u_{n-1} = (E - V_n)u_n$, when the potential is periodic in n, then this leads to a linear recursion problem. For example, if $V_n = V$ is constant, then $u_{n+1} + u_{n-1} = (E - V)u_n$. A question is for which E the solutions stay bounded. You have seen above the case $E - V = 1$.

EXAMPLE 2: Chaos theory. In plasma physics, one studies maps like $(x, y) \mapsto (2x - y - a\sin(x), x)$. You see that $(0, 0)$ is a fixed point. Near that fixed point, the map is described by its linearization $DT(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2-a & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. For which a is this linear system stable near $(0, 0)$ in the sense that a point near $(0, 0)$ stays nearby for all times? The answer will be given using eigenvalues. Note that the matrix here is the same as in the quantum mechanical example before by putting $E = 2 - a$.

EXAMPLE 3: Markov Processes. The percentage of people using Apple OS or the Linux OS is represented by a vector $\begin{bmatrix} m \\ l \end{bmatrix}$. Each cycle 2/3 of Mac OS users switch to Linux and 1/3 stays. Also lets assume that 1/2 of the Linux OS users switch to apple and 1/2 stay. The matrix $P = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$ encoding this dynamics is called a **Markov matrix**: the entries satisfy $0 \leq P_{ij} \leq 1$ and the sum of each column elements is equal to 1. What ratio of Apple/Linux users do we have after things settle to an equilibrium? We can simulate this with a dice: start in a state like $M = (1, 0)$ (all users have Macs). If the dice shows 3,4,5 or 6, a user in that group switch to Linux, otherwise stys in the M camp. Throw also a dice for each user in L. If 1,2 or 3 shows up, the user switches to M. The matrix P has an eigenvector $(3/7, 4/7)$ which belongs to the eigenvalue 1. The interpretation of $P\vec{v} = \vec{v}$ is that with this split up, there is no change in average.