LINEAR DIFFERENTIAL EQUAIONS IN THE PLANE

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REVIEW ROTATION DILATION MATRICES.

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

has the eigenvalue $\lambda_+ = a + ib$ to the eigenvector $v_+ = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and the eigenvalue $\lambda_- = a - ib$ to the eigenvector $v_- = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

For the FIRST ORDER LINEAR DIFFERENTIAL EQUATION

$$\begin{aligned} \frac{d}{dt}x &= ax - by \\ \frac{d}{dt}y &= bx + ay \end{aligned}$$

we have $\frac{d}{dt}\vec{v}_- = (a + ib)\vec{v}_-$ and so

$$\vec{v}_+(t) = e^{at}e^{+ibt}\vec{v}_+$$

Similar, we get

$$\vec{v}_-(t) = e^{at}e^{-ibt}\vec{v}_- \; .$$

With a general initial condition $\vec{x}(0) = c_+\vec{v}_+ + c_-\vec{v}_-$, we have then $\vec{x}(t) = c_+\vec{v}_+(t) + c_-\vec{v}_-(t) = e^{at}(c_+e^{+ibt} + c_-e^{-ibt})$. For $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have $c_+ = 1/2, c_- = 1/2$ we get with that initial condition $\vec{x}(t) = e^{at}(\vec{v}_+(t) + \vec{v}_-(t))/2 = e^{at}\begin{bmatrix} \cos(bt) \\ \sin(bt) \end{bmatrix}$. For $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we have $c_+ = 1/(2i), c+- = -1/(2i)$ so that $\vec{x}(t) = e^{at}(\vec{v}_+(t) - \vec{v}_-(t))/(2i) = e^{at}\begin{bmatrix} -\sin(bt) \\ \cos(bt) \end{bmatrix}$.

The initial value problem

$$\begin{aligned} \frac{d}{dt}x &= ax - by \\ \frac{d}{dt}y &= bx + ay \end{aligned}$$

with $x(0) = A, y(0) = B$ has the explit solutions

$$\vec{x}(t) = e^{at}\begin{bmatrix} A\cos(bt) - B\sin(bt) \\ B\cos(bt) + A\sin(bt) \end{bmatrix}$$

HOW DOES THE PHASE PORTRAIT LOOK LIKE IF $a = 0$?

HOW DOES THE PHASE PORTRAIT LOOK LIKE IF $a < 0$ and $b = 0$?

HOW DOES THE PHASE PORTRAIT LOOK LIKE IF $a > 0$ and $b = 0$?

SKETCH THE PHASE PORTRAIT IN THE CASE $a < 0$ and $b > 0$.

SKETCH THE PHASE PORTRAIT IN THE CASE $a < 0$ and $b < 0$.

SKETCH THE PHASE PORTRAIT IN THE CASE $a > 0$ and $b > 0$.

SKETCH THE PHASE PORTRAIT IN THE CASE $a > 0$ and $b < 0$.