Recall: In this homework, we look at the **inner product space** with

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\ dx\ .$$

One can define length, distances or angles in the same way as we have done with the dot product for vectors in R^n. Functions are assumed to be (piecewise) smooth.

Homework for first lesson (inner product spaces)

1. Find the angle between $f(x) = \cos(x)$ and $g(x) = x^2$. (Like in $\mathbb{R}^n$, we define the angle between f and g to be $\arccos\frac{\langle f,g\rangle}{\|f\|\|g\|}$ where $\|f\| = \sqrt{\langle f, f\rangle}$.)

   Remarks. Use integration by parts twice to compute the integral. This is a good exercise if you feel a bit rusty about integration techniques. Feel free to double check your computation with the computer but try to do the computation by hand.

2. A function on $[-\pi, \pi]$ is called **even** if $f(-x) = f(x)$ for all x and **odd** if $f(-x) = -f(x)$ for all x. For example, $f(x) = \cos x$ is even and $f(x) = \sin x$ is odd.

   a) Verify that if f, g are even functions on $[-\pi, \pi]$, their inner product can be computed by $\langle f, g\rangle = \frac{2}{\pi}\int_0^{\pi} f(x)g(x)\ dx$.

   b) Verify that if f, g are odd functions on $[-\pi, \pi]$, their inner product can be computed by $\langle f, g\rangle = \frac{2}{\pi}\int_0^{\pi} f(x)g(x)\ dx$.

   c) Verify that if f is an even function on $[-\pi, \pi]$ and g is an odd function on $[-\pi, \pi]$, then $\langle f, g\rangle = 0$.

3. Which of the two functions $f(x) = \cos(x)$ or $g(x) = \sin(x)$ is closer to the function $h(x) = x^2$?

4. Determine the projection of the function $f(x) = x^2$ onto the “plane” spanned by the two orthonormal functions $g(x) = \cos(x)$ and $h(x) = \sin(x)$.

   Hint. You have computed the inner product between f and g already in problem 1). Think before you compute the inner product between f and h. There is no calculation necessary to compute $\langle f, h\rangle$.

5. Recall that $\cos(x)$ and $\sin(x)$ are orthonormal. Find the length of $f(x) = a\cos(x) + b\sin(x)$ in terms of a and b.

Homework for second lesson (Fourier series)

1. Find the Fourier series of the function $f(x) = |x|$.

2. Find the Fourier series of the function $\cos^2(x) + 5\sin(x) + 5$. You may find the double angle formula $\cos^2(x) = \frac{\cos(2x)+1}{2}$ useful.

3. Find the Fourier series of the function $f(x) = |\sin(x)|$.

4. In problem 3) you should have gotten a series

   $$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos(2x)}{2^2-1} + \frac{\cos(4x)}{4^2-1} + \frac{\cos(6x)}{6^2-1} + ...\right)$$

   Use Parseval’s identity (Fact 5.5.6 in the book) to find the value of

   $$\frac{1}{(2^2-1)^2} + \frac{1}{(4^2-1)^2} + \frac{1}{(6^2-1)^2} + \cdots$$