JORDAN NORMAL FORM **Math 21b, O. Knill**

JORDAN NORMAL FORM THEOREM.

Every $n \times n$ matrix A is similar to a matrix

$$\begin{bmatrix} [A_1] & 0 & \dots & 0 \\ 0 & [A_2] & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & & [A_k] \end{bmatrix},$$

where A_i are matrices of the form

$$A_i = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

are matrices called **Jordan blocks**

EXAMPLES OF MATRICES IN JORDAN FORM:

1) A generalized shear matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. It is itself a Jordan block.

2) $\begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$. This matrix has three Jordan blocks. The first one is a 2x2 Jordan block, the second a 1x2 Jordan block, the third again a 2x2 Jordan block.

3) Every diagonal matrix like

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

is in Jordan normal form. It consists of 5 Jordan blocks.

4)

$$\begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

is in Jordan normal form. It consists of 4 Jordan blocks. Note that the diagonal elements can be complex and that we can have different blocks with the same diagonal elements. The eigenvalue 5 for example has here 3 Jordan blocks of size 1x1, 2x2 and 3x3.

QUESTION: How many different Jordan normal forms do exists for a 5×5 matrix with eigenvalues 3 and 2 of algebraic multiplicity 3 and 2?

ANSWER: Examples 2) and 3) are examples. There are more.

QUESTION: Is $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$. in Jordan normal form?