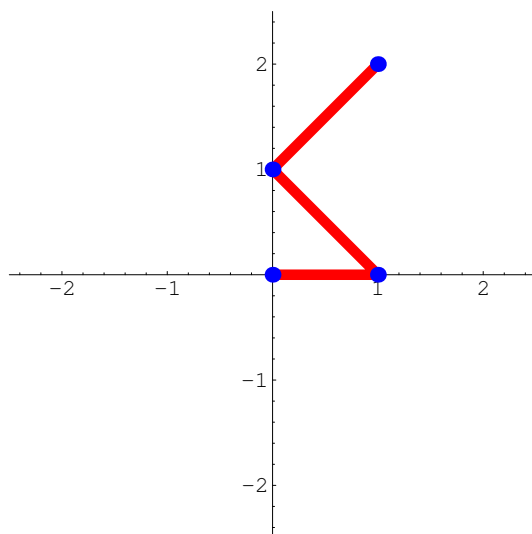


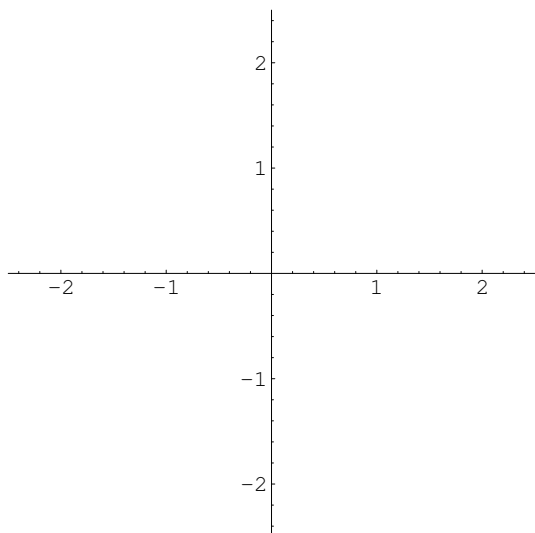
Linear Transformations in *Geometry*

Consider the following figure, which has endpoints $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

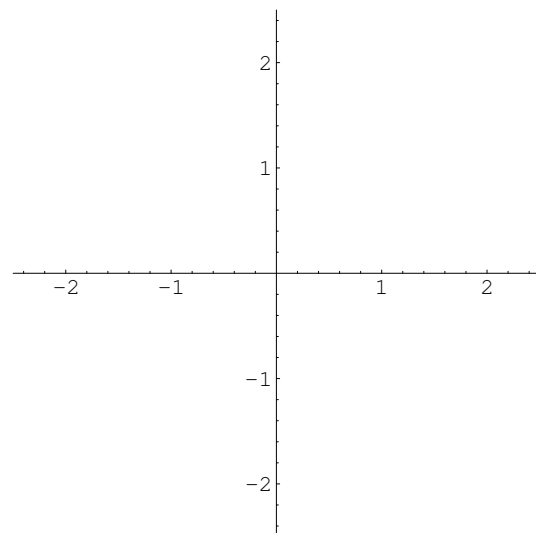


Draw what happens to the figure after applying the linear transformation $T(\vec{x}) = A\vec{x}$ in each of the following cases. Try to describe the effect of the linear transformation in words. Is the linear transformation invertible?

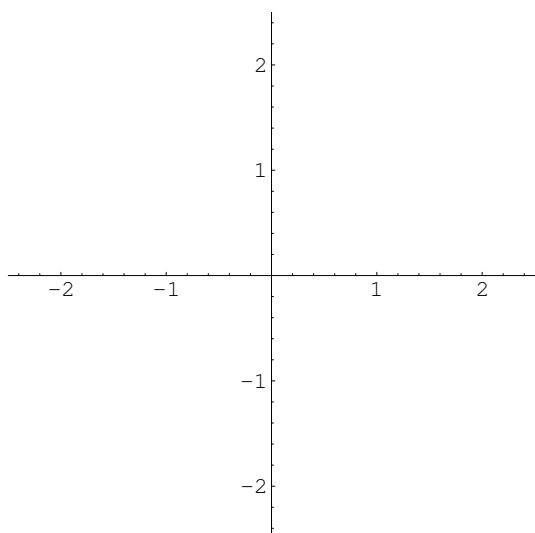
1. $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$



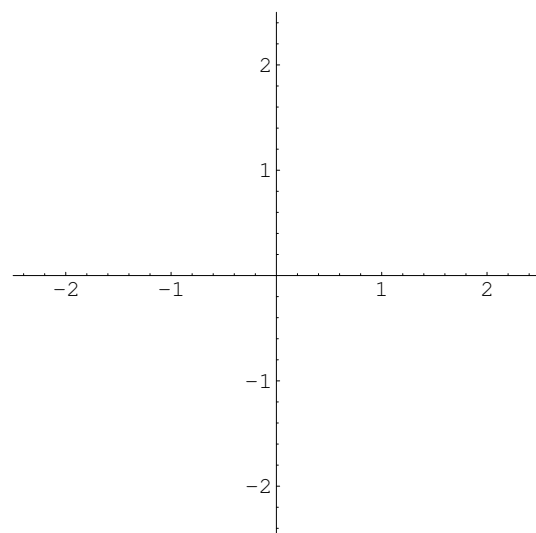
2. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$



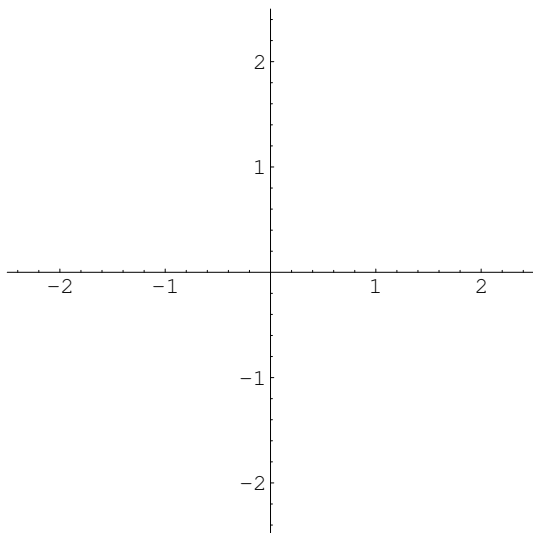
3. $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$



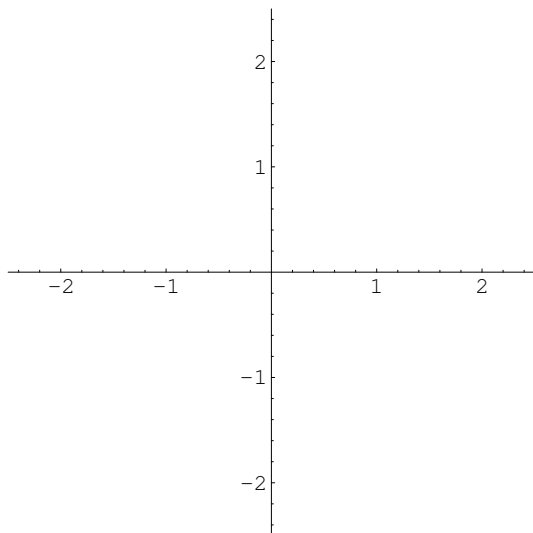
4. $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$



5. $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$



6. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



Invertible Matrices

Let A be an $n \times m$ matrix (n rows and m columns). Remember that A is said to be invertible if the equation $A\vec{x} = \vec{y}$ has a unique solution for every $\vec{y} \in \mathbb{R}^n$.

1. Suppose $\text{rank}(A) < n$. Is A invertible? That is, is it true that the system $A\vec{x} = \vec{y}$ has a unique solution for every $\vec{y} \in \mathbb{R}^n$? (Hint: what does $\text{rref}(A)$ look like?)

2. Suppose $\text{rank}(A) < m$. Is A invertible?

3. Finally, suppose $\text{rank}(A) = n$ and $\text{rank}(A) = m$ (in particular, $n = m$). Is A invertible?

An Example of Finding Inverses

Remember that, to find the inverse of a matrix A , we want to solve the system $A\vec{x} = \vec{y}$ to write $\vec{x}$ in terms of $\vec{y}$. As an example, consider the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}.$$

If we write

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

then the equation $A\vec{x} = \vec{y}$ can be rewritten as

$$\left| \begin{array}{ccc} 3x_1 & + & 2x_2 & = & y_1 \\ 7x_1 & + & 5x_2 & = & y_2 \end{array} \right|.$$

To solve this, we just use the row operations we know (and love!):

$$\begin{aligned} \left| \begin{array}{ccc} 3x_1 & + & 2x_2 & = & y_1 \\ 7x_1 & + & 5x_2 & = & y_2 \end{array} \right| \div 3 &\rightarrow \left| \begin{array}{ccc} x_1 & + & \frac{2}{3}x_2 & = & \frac{1}{3}y_1 \\ 7x_1 & + & 5x_2 & = & y_2 \end{array} \right| -7(\text{I}) \\ &\rightarrow \left| \begin{array}{ccc} x_1 & + & \frac{2}{3}x_2 & = & \frac{1}{3}y_1 \\ \frac{1}{3}x_2 & = & -\frac{7}{3}y_1 & + & y_2 \end{array} \right| \times 3 \\ &\rightarrow \left| \begin{array}{ccc} x_1 & + & \frac{2}{3}x_2 & = & \frac{1}{3}y_1 \\ & & x_2 & = & -7y_1 + 3y_2 \end{array} \right| -\frac{2}{3}(\text{II}) \\ &\rightarrow \left| \begin{array}{ccc} x_1 & & & = & 5y_1 - 2y_2 \\ & & x_2 & = & -7y_1 + 3y_2 \end{array} \right| \end{aligned}$$

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

so

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}.$$

Matrix Multiplication Practice

1. Let $A = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$. Find AB and BA (if they make sense).

2. Let $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find AB and BA (if they make sense).

3. Let $A = \begin{bmatrix} 1 & 4 \\ -3 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}$. Find AB and BA (if they make sense).

4. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Find AB , $(AB)^{-1}$, and $A^{-1}B^{-1}$.

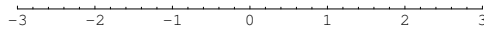
Answers are at <http://abel.math.harvard.edu/~jjchen/math21b/>

Solutions of Systems

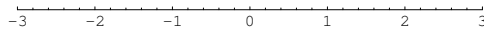
1. Define a linear transformation T from $\mathbb{R}$ to $\mathbb{R}^2$ by

$$T(x) = \begin{bmatrix} 3x \\ 2x \end{bmatrix}.$$

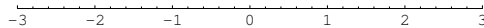
- (a) Draw the set of $x \in \mathbb{R}$ such that $T(x) = \vec{0}$. (This set is called the kernel of T .)



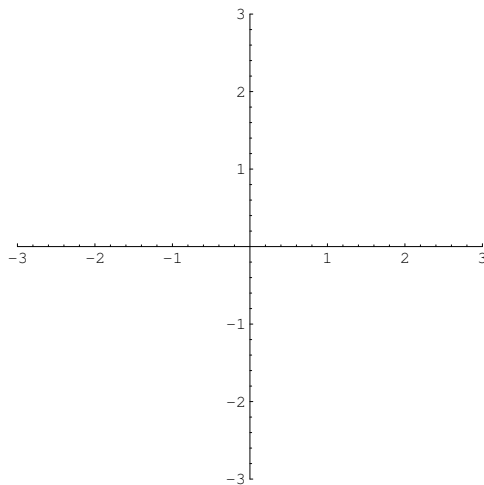
- (b) Draw the set of $x \in \mathbb{R}$ such that $T(x) = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$.



- (c) Draw the set of $x \in \mathbb{R}$ such that $T(x) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

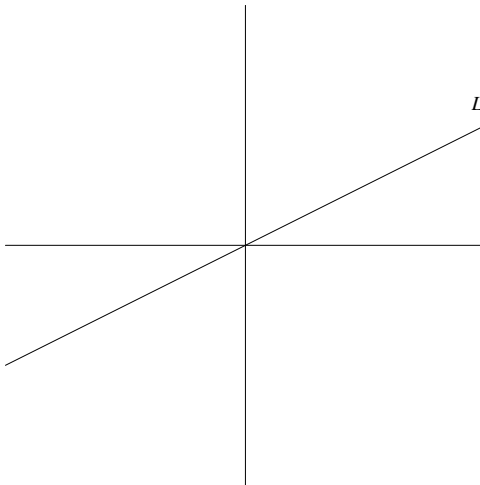


- (d) Draw the set of $\vec{y} \in \mathbb{R}^2$ such that $T(x) = \vec{y}$ has at least one solution. (This set is called the image of T .)

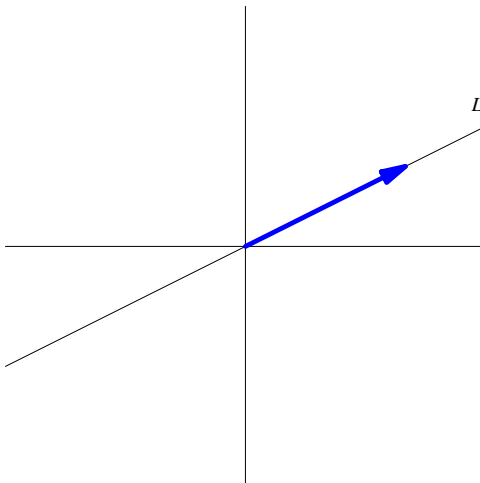


2. Let L be a line through the origin. Remember that we defined a linear transformation proj_L from $\mathbb{R}^2$ to $\mathbb{R}^2$.

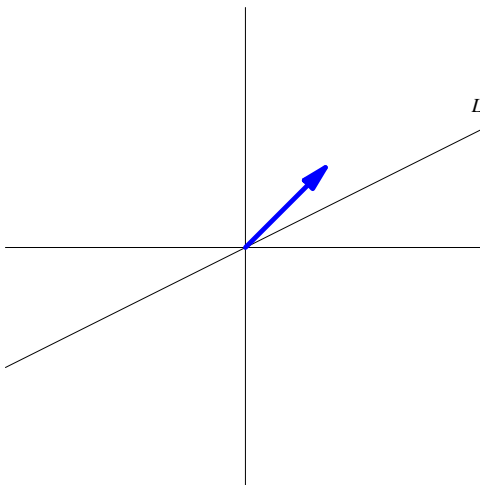
- (a) Draw the set of $\vec{x} \in \mathbb{R}^2$ such that $\text{proj}_L(\vec{x}) = \vec{0}$. (This is the kernel of proj_L .)



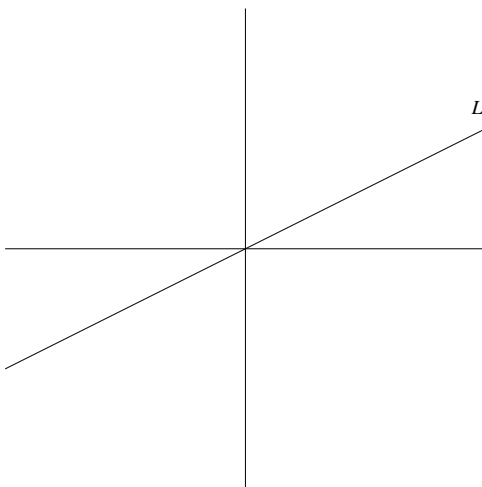
- (b) In the picture below, a vector $\vec{y}$ is drawn. Draw the set of $\vec{x} \in \mathbb{R}^2$ such that $\text{proj}_L(\vec{x}) = \vec{y}$.



- (c) In the picture below, a vector $\vec{y}$ is drawn. Draw the set of $\vec{x} \in \mathbb{R}^2$ such that $\text{proj}_L(\vec{x}) = \vec{y}$.



- (d) Draw the set of $\vec{y} \in \mathbb{R}^2$ such that $\text{proj}_L(\vec{x}) = \vec{y}$ has at least one solution. (This is the image of proj_L .)



3. If T is any linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ and $\vec{y}$ is any vector in $\mathbb{R}^n$, how does the solution set of $T(\vec{x}) = \vec{y}$ relate to the solution set of $T(\vec{x}) = \vec{0}$?

Subspace and Basis Examples

1. Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Is the set of solutions of $A\vec{x} = \vec{b}$ a subspace of $\mathbb{R}^3$?

2. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$. Are any of these vectors redundant?

3. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 7 \\ 1 & 2 & 4 \end{bmatrix}$. Find a basis of $\text{im } A$.

4. Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_6$ be vectors in $\mathbb{R}^8$ such that $3\vec{v}_1 - 2\vec{v}_2 + 4\vec{v}_4 + 5\vec{v}_5 = \vec{0}$. Explain why $\vec{v}_1, \dots, \vec{v}_6$ must be linearly dependent.

5. Let V be a subspace of $\mathbb{R}^n$, and suppose we know that $\vec{v}_1, \dots, \vec{v}_5$ form a basis of V . Is it possible that $\vec{v}_1 + 2\vec{v}_2 - \vec{v}_4 = \vec{v}_3 + 2\vec{v}_4 + \vec{v}_5$?

6. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 7 \\ 1 & 2 & 4 \end{bmatrix}$, like in Problem 3. Find $\ker A$. How can you use your answer to determine whether the columns of A are linearly independent?

The Rank - Nullity Theorem

Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 2 & 1 & 4 & -1 & 2 \\ -1 & 2 & -7 & 0 & 4 \\ 3 & 0 & 9 & -1 & -1 \end{bmatrix}$. You (should) know by now how to find $\text{rref}(A)$, so I'll just tell you what it is: $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & 0 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Let $\vec{v}_1, \dots, \vec{v}_5$ be the columns of A .

1. Find a basis of $\ker A$. What is the dimension of $\ker A$? How does the dimension of $\ker A$ relate to the rank of A ? (*Note:* the dimension of $\ker A$ is called the *nullity* of A .)

2. For each basis vector $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$ that you found in problem #1:

(a) Find $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 + c_5\vec{v}_5$. (Why does your answer make sense?)

(b) Use your answer from #2b to find a redundant vector among $\vec{v}_1, \dots, \vec{v}_5$.

3. After throwing away the redundant vectors you found in #2, are the remaining $\vec{v}_i$ linearly independent? Why or why not?
4. Find a basis of $\text{im } A$. What is the dimension of $\text{im } A$? How does the dimension of $\text{im } A$ relate to the rank of A ?

Coordinates

1. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$. Then, $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ is a basis of $\mathbb{R}^2$. Let L be the line $y = 2x$.

(a) Find $[\vec{e}_1]_{\mathfrak{B}}$.

(b) Find a matrix S such that $\vec{x} = S[\vec{x}]_{\mathfrak{B}}$ for all $\vec{x} \in \mathbb{R}^2$.

(c) Find the $\mathfrak{B}$ -matrix of proj_L .

(d) Use the $\mathfrak{B}$ -matrix of proj_L to find the standard matrix A of proj_L .

2. Let $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then, $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ is a basis of $\mathbb{R}^2$. Let $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$ and $T(\vec{x}) = A\vec{x}$. Find the matrix of T with respect to the basis $\mathfrak{B}$.

3. Let P be the plane $x_1 + 2x_2 - 3x_3 = 0$. Any vector $\vec{x} \in \mathbb{R}^3$ can be written uniquely as a sum $\vec{x}^P + \vec{x}^\perp$ where $\vec{x}^P$ is a vector in the plane P and $\vec{x}^\perp$ is a vector perpendicular to P . We define a linear transformation proj_P from $\mathbb{R}^3$ to $\mathbb{R}^3$ by $\text{proj}_P(\vec{x}) = \vec{x}^P$. (This is very much like the projection transformations we talked about before, except that we are now projecting onto a plane rather than a line.)

(a) Find a convenient basis $\mathfrak{B}$ of $\mathbb{R}^3$ and write the $\mathfrak{B}$ -matrix of proj_P .

(b) Use your answer from part (a) to find the standard matrix of proj_P .

Linear Spaces

1. In each problem, decide if the set is a linear space. If it is, find the neutral element.

(a) The set of 2×2 matrices with positive entries.

(b) $\mathbb{R}^{n \times m}$, the set of $n \times m$ matrices.

(c) The set of polynomials of the form $x^2 + bx + c$ where $b, c \in \mathbb{R}$.

(d) $F(\mathbb{R}, \mathbb{R})$, the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

2. Which of the following are subspaces of $\mathbb{R}^{2 \times 2}$?

(a) The 2×2 matrices of rank 1.

(b) The 2×2 matrices of rank less than or equal to 1.

(c) The 2×2 matrices of rank 0.

3. Which of the following are subspaces of $F(\mathbb{R}, \mathbb{R})$?

(a) The set of all functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $f(0) = 0$.

(b) The set of all functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $f(0) = 0$ and $f(1) \geq 0$.

4. (a) Find a basis $\mathfrak{B}$ of $\mathbb{R}^{2 \times 2}$, and write the matrix $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ in $\mathfrak{B}$ -coordinates.

(b) Find a basis $\mathfrak{B}$ of P_4 , and write the polynomial $x^4 - x^2 + 3$ in $\mathfrak{B}$ -coordinates.

5. Find the dimension of $P_n = \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 : a_i \in \mathbb{R}\}$, the set of polynomials of degree at most n . What can you say about the dimension of P , the set of all polynomials?

The Gram-Schmidt Process

Let $M = \begin{bmatrix} 1 & 3 & 1 \\ 1 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ and $V = \text{im } M$. Then, the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}$$

form a basis of V (do you see why?).

1. Find an orthonormal basis of V .

2. Find the QR -factorization of M .

3. Find the QR -factorization of $M = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$.

Least Squares and Data Fitting

1. Suppose we want to find a quadratic polynomial $f(x) = ax^2 + bx + c$ such that $f(-1) = 1$, $f(0) = 0$, $f(1) = 2$, and $f(2) = 5$. Express this problem as a linear system.

2. We want to fit a linear function of the form $f(x) = mx + c$ to the data points $(-1, 3)$, $(0, 1)$, and $(1, 1)$.

(a) Do you expect m to be positive, negative, or zero?

(b) Find the linear function $f(x) = mx + c$.

3. The following table describes the percent of classes that Harvard students attend:

Year (y)	Percent of Classes Attended (p)
1 (Freshman)	100
2 (Sophomore)	90
3 (Junior)	60
4 (Senior)	10

We suspect that $p(y)$ looks like ky^n for some constants k and n , and we would like to find k and n .

(a) Do you expect k and n to be positive or negative? What number should k be close to?

(b) Express this problem as a linear system.

Odds and Ends

1. If A is an $n \times m$ matrix, what is the relationship between $\text{rank}(A^T)$ and $\text{rank}(A)$?

2. In each problem, decide if S is a subspace of $\mathbb{R}^{3 \times 3}$, the linear space of 3×3 matrices. If it is, find the dimension of S .
 - (a) S is the set of symmetric 3×3 matrices.

 - (b) S is the set of skew-symmetric 3×3 matrices.

 - (c) S is the set of orthogonal 3×3 matrices.

3. Explain why the matrix of a reflection must be symmetric.

More on Determinants

$$\text{Let } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 5 & 8 & 6 & 7 \\ 9 & 12 & 10 & 11 \end{bmatrix}.$$

1. Write an expression for $\det A$ in terms of determinants of 2×2 matrices.

2. Let B be A with the third and fourth rows swapped. That is,

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 9 & 12 & 10 & 11 \\ 5 & 8 & 6 & 7 \end{bmatrix}.$$

Write an expression for $\det B$ in terms of determinants of 2×2 matrices. How does this relate to your expression for $\det A$?

3. Let B be A with a multiple of the fourth row added to the third row. That is,

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 9k+5 & 12k+8 & 10k+6 & 11k+7 \\ 9 & 12 & 10 & 11 \end{bmatrix}.$$

Write an expression for $\det B$ in terms of determinants of 2×2 matrices. How does this relate to your expression for $\det A$?

4. Let $A = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 3 \\ 2 & -2 & -2 \end{bmatrix}$. Use row reduction to find $\det A$.

5. If A is an orthogonal $n \times n$ matrix, what can you say about $\det A$?

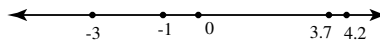
6. If A is any $n \times m$ matrix with linearly independent columns, A has a QR -factorization $A = QR$. Express $\det(A^T A)$ in terms of $\det R$. (Although A is not necessarily square, $A^T A$ is always square, so $\det(A^T A)$ makes sense.)

Complex Numbers



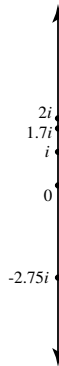
Basic Definitions

You're used to real numbers and how they're graphed on the real number line.



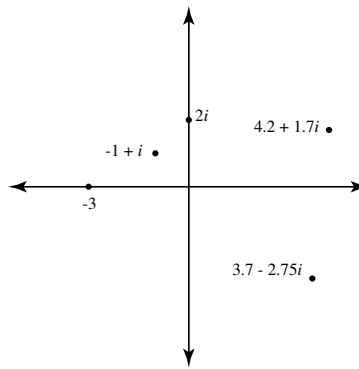
Real numbers are very useful, but there is one major problem we encounter when working with them: not all polynomials have real roots. For example, although the polynomials $x^2 - 3$ and $x^2 + 1$ look almost identical, the former has two real roots ($\pm\sqrt{3}$) while the latter has none. As we saw in class, this means that some matrices don't have any real eigenvalues. To fix this problem, mathematicians simply define a new number, called i , to be the square root of -1 . Of course, i is not a real number, since there is no real number whose square is -1 ; instead, we call i an imaginary number. Now, the polynomial $x^2 + 1$ has two roots, i and $-i$.

More generally, an imaginary number is any real number multiple of i , like 0 , $2i$, $-2.75i$, or $1.7i$. We can graph the imaginary numbers on a number line, but we use a vertical line instead of a horizontal line.



We know how to add and multiply real numbers, and we would like to do the same with imaginary numbers. However, if two imaginary numbers are multiplied, the answer is a real number; for instance, $(2i)(3i) = 6i^2 = -6$. This suggests that we shouldn't look at real and imaginary numbers separately; instead, we study complex numbers, which are sums of real and imaginary numbers. That is, a complex number is just a number of the form $x + iy$ where x and y are both real numbers. So, $1 + \sqrt{3}i$, $4.2 + 1.7i$, -3 , $3.7 - 2.75i$, and $2i$ are all complex numbers. We write $\mathbb{C}$ for the set of complex numbers.

Since we view the set of real numbers as a horizontal line and the set of imaginary numbers as a vertical line, it's natural to view the set of complex numbers as a plane in which the real numbers lie on the horizontal axis and the imaginary numbers lie on the vertical axis. This plane is called the complex plane.



Arithmetic of Complex Numbers

As we said already, a complex number is just a number of the form $z = x + iy$ where $x, y \in \mathbb{R}$. We call x the real part of z and y the imaginary part of z . To add complex numbers, we just add the real and imaginary parts separately. That is, $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$.

Example 1. $(4.2 + 1.7i) + (3.7 - 2.75i) = (4.2 + 3.7) + (1.7 - 2.75)i = 7.9 - 1.05i$. ❖

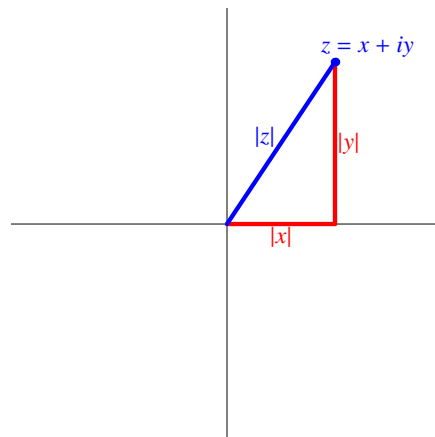
To multiply complex numbers, we use the distributive property. That is,

$$\begin{aligned}
 (x_1 + iy_1)(x_2 + iy_2) &= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2) \\
 &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 \\
 &= x_1x_2 + ix_1y_2 + iy_1x_2 - y_1y_2 \text{ since } i^2 = -1 \\
 &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)
 \end{aligned}$$

Example 2. $(1 + i)(5 + 3i) = 2 + 8i$. ❖

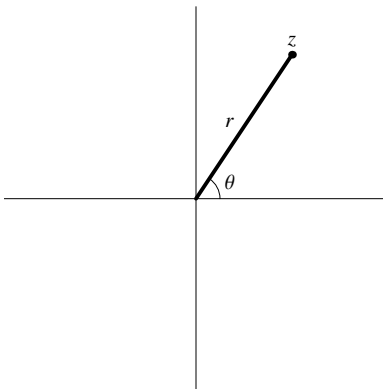
The Absolute Value of a Complex Number

When x is a real number, the absolute value of x measures the distance from x to 0 on the number line. For instance, 5 and -5 are both 5 units away from 0, so $|5| = |-5| = 5$. Similarly, the absolute value of a complex number $z = x + iy$ is defined to be the distance from z to 0. By the Pythagorean theorem, $|z| = \sqrt{x^2 + y^2}$.



Polar Coordinates

When we write a complex number z as $x + iy$ with $x, y \in \mathbb{R}$, we say that we are writing z in Cartesian coordinates. There is another useful way to write a complex number: any complex number z is determined by its distance from the origin (r) and an angle (θ):

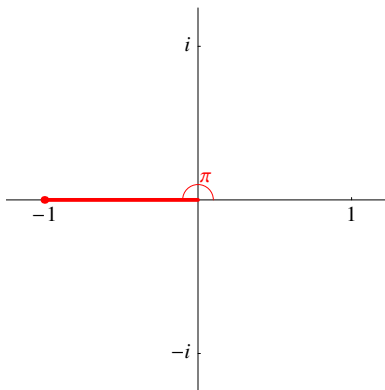


We call r and θ the polar coordinates of z . We can see from the diagram that $z = r \cos \theta + i(r \sin \theta)$. We write this more simply as $z = re^{i\theta}$. (If you want to know why $e^{i\theta} = \cos \theta + i \sin \theta$, try writing out the Taylor series of e^x , $\sin x$, and $\cos x$.)

To summarize, we can write a complex number z in the form $z = x + iy$ (Cartesian coordinates) or the form $z = re^{i\theta}$ (polar coordinates). These two forms are related as follows.

- $x = r \cos \theta$ and $y = r \sin \theta$.
- $r = |z| = \sqrt{x^2 + y^2}$ and $\tan \theta = \frac{y}{x}$.

Example 3. $e^{\pi i}$ is the complex number with $r = 1$ and $\theta = \pi$. By drawing a diagram, we see that $e^{\pi i} = -1$.



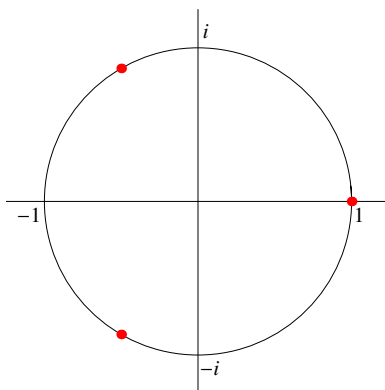
Similarly, $e^{2\pi i} = 1$. Notice that, if n is any integer, then $e^{2\pi in} = 1$. ❖

If we are adding complex numbers, it is useful to write them in Cartesian coordinates; on the other hand, if we are multiplying complex numbers, it is often easier to use polar coordinates. After all, $(r_1 e^{i\theta_1})(r_2 e^{i\theta_2})$ is just $r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$. Here are two examples of how polar coordinates can be useful.

Example 4. Let $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Suppose we want to find z^{100} . We could just start computing powers of z , but that would get boring really fast. Instead, let's write z in polar coordinates. We know that $z = re^{i\theta}$ where $r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$ and $\tan \theta = \sqrt{3}$. Thus, θ is either $\frac{\pi}{3}$ or $\frac{4\pi}{3}$. Since z lies in the first quadrant

of the complex plane, it must be the case that $\theta = \frac{\pi}{3}$, so $z = e^{\pi i/3}$. Then, $z^{100} = (e^{\pi i/3})^{100} = e^{100\pi i/3}$. In Cartesian coordinates, $z^{100} = \cos \frac{100\pi}{3} + i \sin \frac{100\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. ♦

Example 5. Let's find all complex numbers z such that $z^3 = 1$. We know we can write any complex number z as $re^{i\theta}$ for some r and θ . Then, $z^3 = r^3 e^{3i\theta}$, so $z^3 = 1$ if and only if $r^3 = 1$ and 3θ is a multiple of 2π . So, the solutions of $z^3 = 1$ are $z = 1$, $e^{2\pi i/3}$, and $e^{4\pi i/3}$. These lie on a circle of radius 1 in the complex plane:



In Cartesian coordinates, $e^{2\pi i/3} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $e^{4\pi i/3} = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. ♦

The Fundamental Theorem of Algebra

As we discussed earlier, not all polynomials have real roots, so not all matrices have real eigenvalues. For complex numbers, the situation is much better.

The Fundamental Theorem of Algebra. *Let $p(x)$ be a polynomial whose coefficients are complex numbers (or just real numbers), and let n be the degree of $p(x)$. Then, $p(x)$ factors as $p(x) = c(x - a_1) \cdots (x - a_n)$ for some complex numbers $c, a_1, \dots, a_n$. In particular, $p(x)$ has exactly n roots $a_1, \dots, a_n$ (if the roots are counted with multiplicity).*

Example 6. The polynomial $x^2 + 1$ can be written as $(x - i)(x + i)$. That is, it has two roots, i and $-i$. ♦

Example 7. The polynomial $x^2 + 2ix - 1$ can be written as $(x + i)^2$, so $-i$ is a root of multiplicity 2. ♦

The Fundamental Theorem of Algebra tells us that every $n \times n$ matrix has *exactly* n complex eigenvalues. Don't worry if you don't understand the theorem perfectly; we will talk about it more in class.

Practice Problems

1. Let $z = \frac{1}{2} + \frac{1}{2}i$. Write z in polar coordinates. Draw $z, z^2, z^3, \dots, z^8$ in the complex plane. Write z^5 in Cartesian coordinates.
2. Explain why $\mathbb{C}$ is a linear space. What is its dimension?
3. If $z = re^{i\theta}$ is a nonzero complex number, write $\frac{1}{z}$ in polar coordinates.
4. Find the square roots of i .

Julia Sets (Optional but Super Cool!)

In class, we talked about a dynamical system involving wolves and deer. Essentially, we came up with a matrix A that represented how wolf and deer populations change over time. Then, if T was the linear transformation $T(\vec{x}) = A\vec{x}$ and $\vec{x}_0$ was a vector representing the starting populations, we found:

- the populations after one year are given by $\vec{x}_1 = T(\vec{x}_0)$,
- the populations after two years are given by $\vec{x}_2 = T(\vec{x}_1)$,
- and so on.

We might try doing the same thing with a different type of transformation; it turns out that, if we make T a quadratic function, we get very interesting objects called Julia sets. You won't need to know about Julia sets for this class, but they are pretty amazing things.

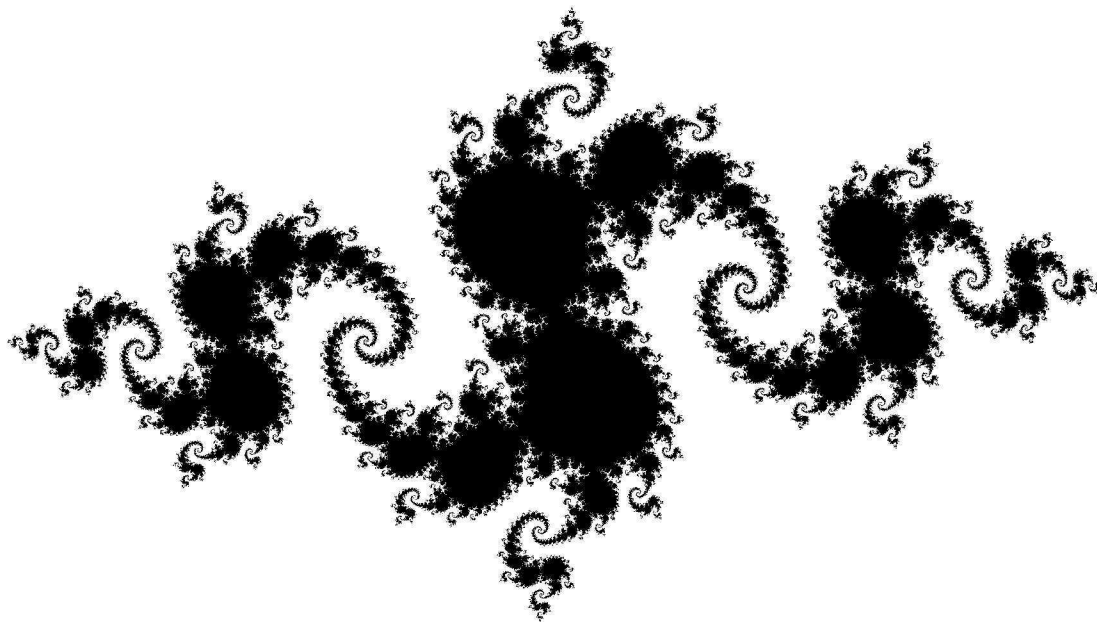
The way to make a Julia set is pretty easy. We start by picking a complex number c , like $c = -0.8 - 0.15i$ (every choice of c gives a different Julia set). Then, we define a function $f(z) = z^2 + c$. If z_0 is any complex number, we define:

$$\begin{aligned} z_1 &= f(z_0) \\ z_2 &= f(z_1) \\ z_3 &= f(z_2) \\ &\vdots \end{aligned}$$

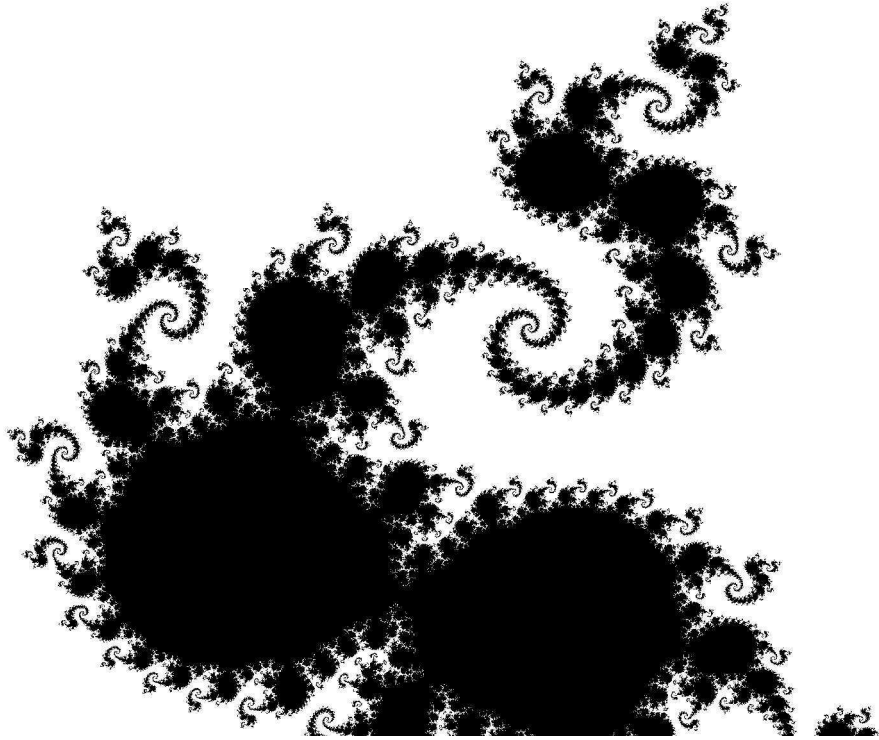
This is just a list of complex numbers $z_0, z_1, z_2, \dots$; we call this list the orbit of z_0 . In general, such orbits may be really wacky; rather than trying to understand them very deeply, we just ask a simple question: do the z_n stay close to the origin? There are two possibilities:

1. Every z_n satisfies $|z_n| < 2$. That is, all elements of the orbit stay within 2 units of the origin. (We say the orbit “does not escape.”)
2. Some z_n satisfies $|z_n| \geq 2$. (We say the orbit “escapes.”)

Now, we are ready to define the Julia set (for c) — it is simply the set of all complex numbers z_0 such that the orbit of z_0 does not escape. Although this definition is relatively straightforward, Julia sets are extremely complicated objects. For example, here is the Julia set for $c = -0.8 - 0.15i$; as you can see, it is not simple at all!



Julia sets have lots of interesting properties. One of the most striking is that tiny pieces of a Julia set look identical to the entire Julia set. For instance, here is a teeny piece of the previous image, magnified about a million times:



The pictures get even more interesting if you color the white points somehow; you can see some examples at <http://math.harvard.edu/~jjchen/fractals/>

Eigenvectors

1. True or false: if $(\vec{v}_1, \dots, \vec{v}_n)$ is an eigenbasis for A , then $(\vec{v}_1, \dots, \vec{v}_n)$ is an eigenbasis for A^2 .
2. Let V be a plane in $\mathbb{R}^3$ which contains the origin, and let A be the matrix of proj_V (that is, $\text{proj}_V(\vec{x}) = A\vec{x}$). Find the eigenvalues of A , and describe the eigenspaces geometrically. Is there an eigenbasis for A ?
3. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, which represents a shear. Find the eigenvalues of A and their algebraic and geometric multiplicities. Is there an eigenbasis for A ?

4. Let A be a noninvertible $n \times n$ matrix. Explain why 0 must be an eigenvalue of A . Find the geometric multiplicity of the eigenvalue 0 in terms of $\text{rank}(A)$.

5. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then, A and B are similar. Find the characteristic polynomials, eigenvalues, and eigenvectors of A and B .

6. Is the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ similar to the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$?

Diagonalization

Let L be the line $y = 2x$ in $\mathbb{R}^2$. Let ref_L be reflection over L , and let A be the standard matrix of ref_L .

1. Find an eigenbasis $\mathfrak{B}$ for A .

2. Find the $\mathfrak{B}$ -matrix of ref_L .

3. Find A (the standard matrix of ref_L).

True / False

1. If A is diagonalizable, then A^2 is diagonalizable.
2. If A and B are $n \times n$ diagonalizable matrices, then $A + B$ is diagonalizable.
3. If A and B are $n \times n$ diagonalizable matrices with the same eigenvectors, then AB is diagonalizable.
4. If A is diagonalizable, then A^T is diagonalizable.
5. If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.
6. If A is a diagonalizable matrix and λ is an eigenvalue of A , then the algebraic multiplicity of λ is equal to the geometric multiplicity of λ .
7. If A and B are both diagonalizable and if A and B have the same eigenvalues with the same geometric multiplicities, then A is similar to B .

Stability

Let A be the matrix $\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$. You are given the following information:

- $\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigenvector with eigenvalue $\frac{1+i}{2}$.
 - $\vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an eigenvector with eigenvalue $\frac{1-i}{2}$.
1. Write A as SDS^{-1} where S is an invertible matrix and D is a diagonal matrix (both with complex entries).
 2. Find a closed formula for A^t .

3. If $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, what do the trajectories of the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ look like? What is the long-term behavior of $\vec{x}(t)$? What if $\vec{x}(0)$ is some other vector?

4. If B is any 2×2 matrix with the same eigenvalues as A , what do the trajectories of $\vec{x}(t+1) = B\vec{x}(t)$ look like?

Symmetric Matrices

1. If A is orthogonally diagonalizable, what is the relationship between A^T and A ?

2. (a) Let A be a matrix, $\vec{v}$ be any vector, and $\vec{w}$ be an eigenvector of A with eigenvalue μ . What is the relationship between $\vec{v}^T A \vec{w}$ and $\vec{v} \cdot \vec{w}$?

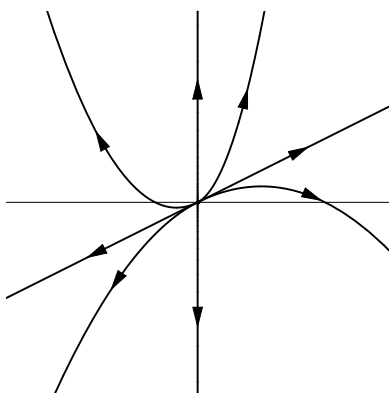
- (b) Let A be a matrix, $\vec{w}$ be any vector, and $\vec{v}$ be an eigenvector of A^T with eigenvalue λ . What is the relationship between $\vec{v}^T A \vec{w}$ and $\vec{v} \cdot \vec{w}$?

- (c) If A is a symmetric matrix and $\vec{v}$ and $\vec{w}$ are eigenvectors with different eigenvalues, what can you say about $\vec{v} \cdot \vec{w}$?

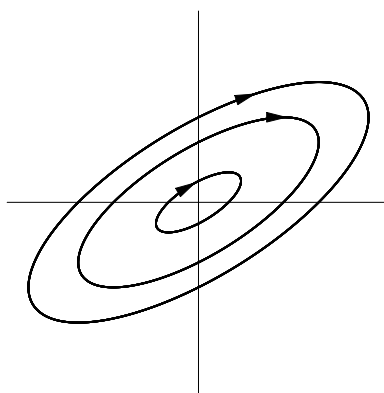
Phase Portraits

Each of the following is the phase portrait of a continuous dynamical system $\frac{d\vec{x}}{dt} = A\vec{x}$ where A is a real 2×2 matrix. What can you say about the eigenvalues, determinant, and trace of A ? In which cases is $\vec{0}$ an asymptotically stable equilibrium?

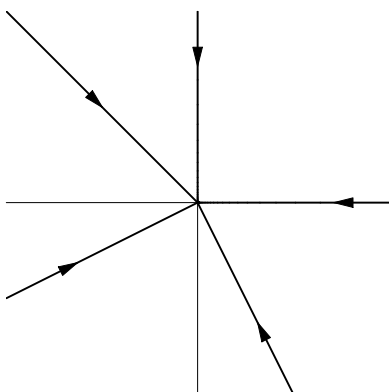
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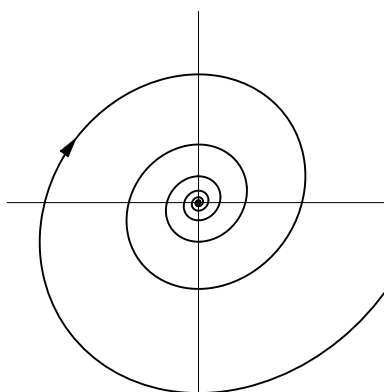
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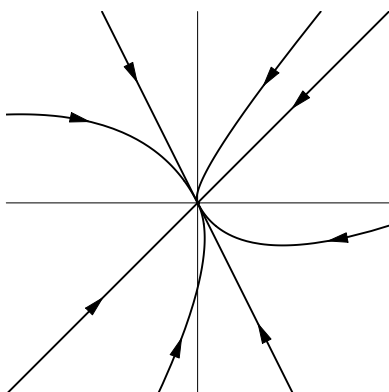
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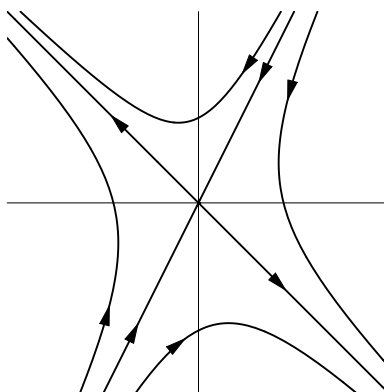
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5.



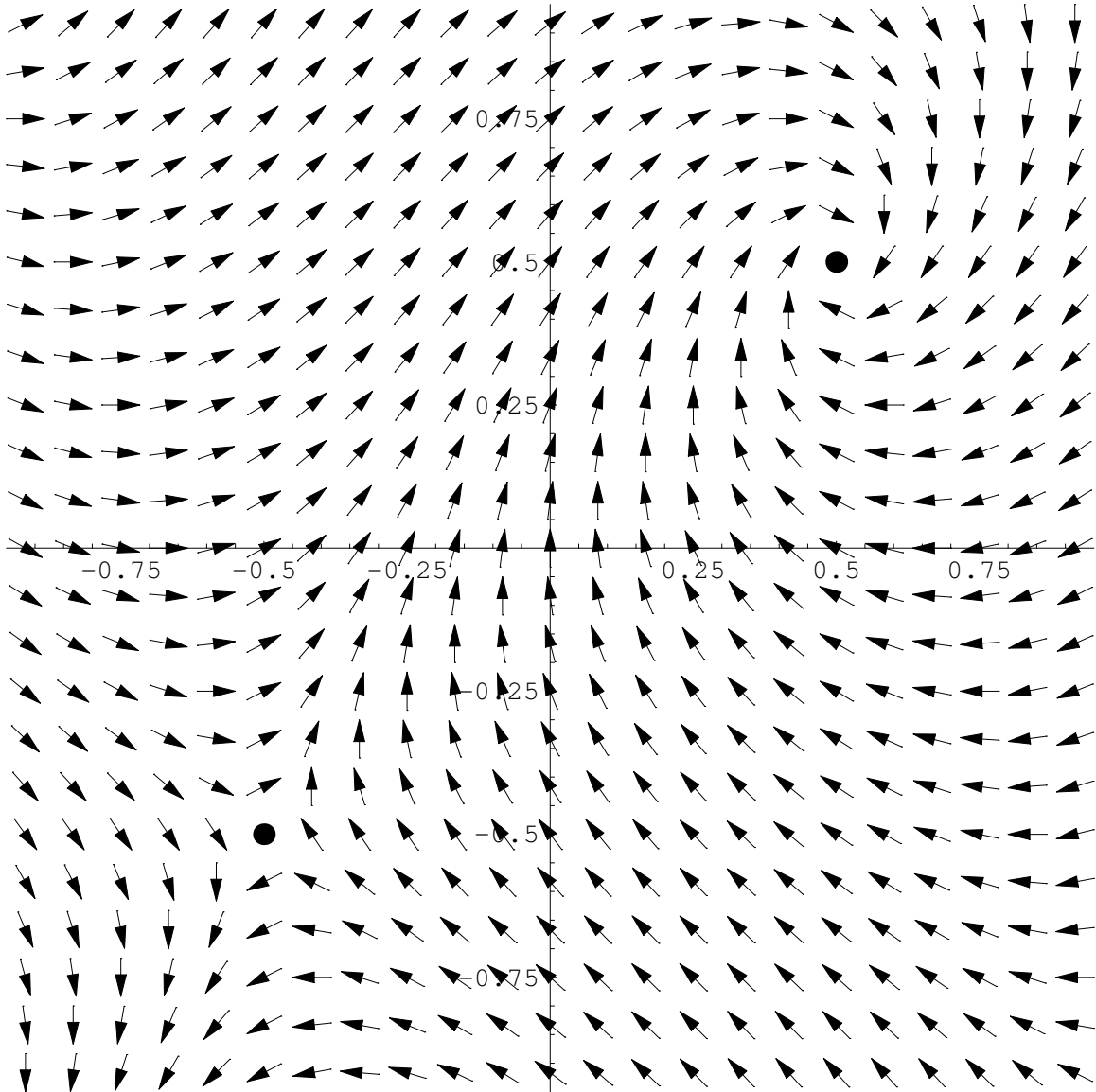
6.



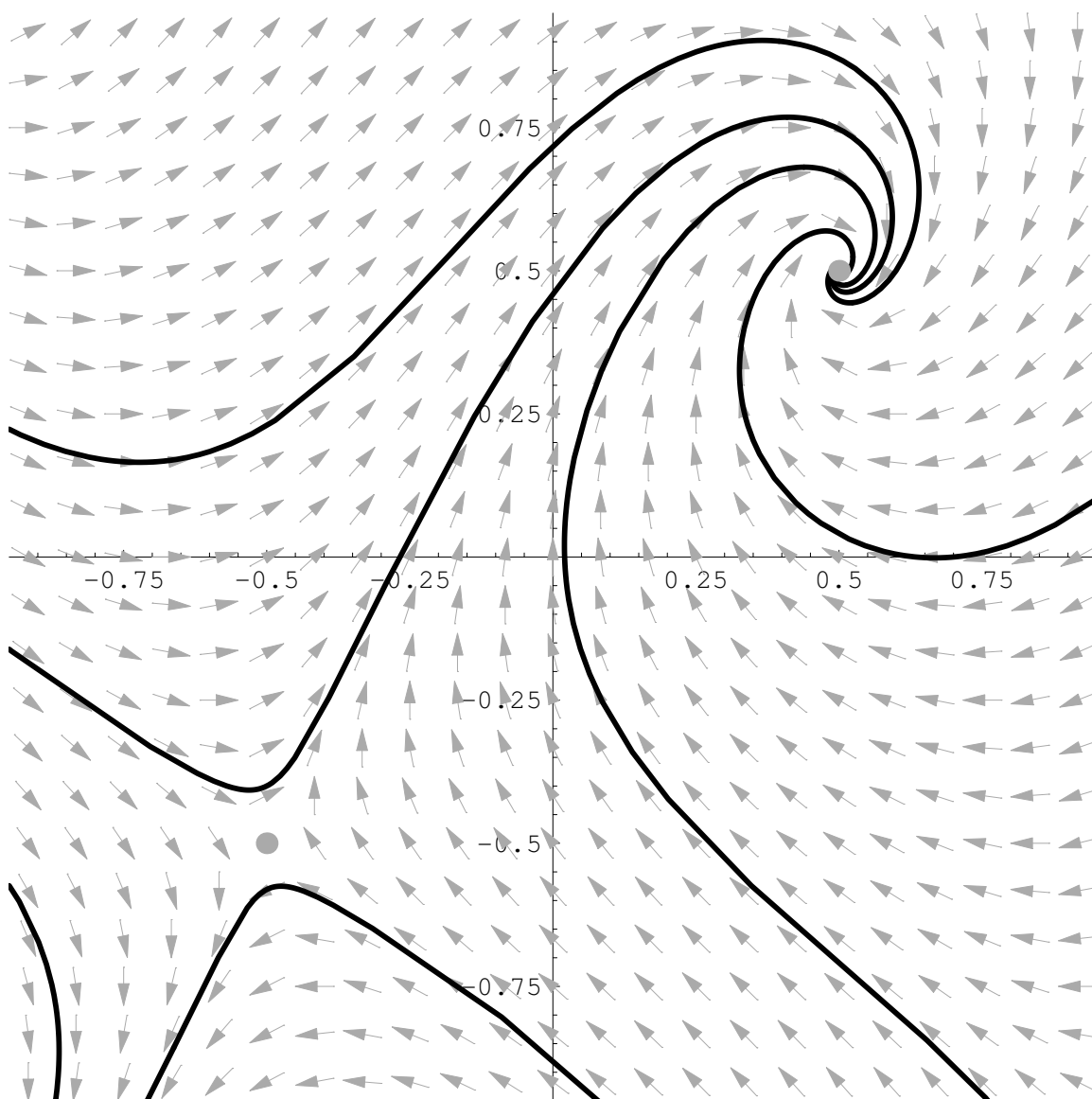
Nonlinear Dynamical Systems

Here is the direction field for the nonlinear system

$$\begin{aligned}\frac{dx}{dt} &= \sin \frac{\pi(y-x)}{2} \\ \frac{dy}{dt} &= \cos \frac{\pi(y+3x)}{4}\end{aligned}$$



Here is the direction field with some trajectories.



Linear Transformations

Which of the following are linear spaces over $\mathbb{R}$? Over $\mathbb{C}$?

1. The set P of polynomials with real coefficients.

2. $V = \{f \in C^\infty(\mathbb{R}) : f(1) = i\}$.

3. The complex numbers $\mathbb{C}$.

Which of the following are linear transformations?

4. D from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$ defined by $D(f) = f'$ (over $\mathbb{C}$).

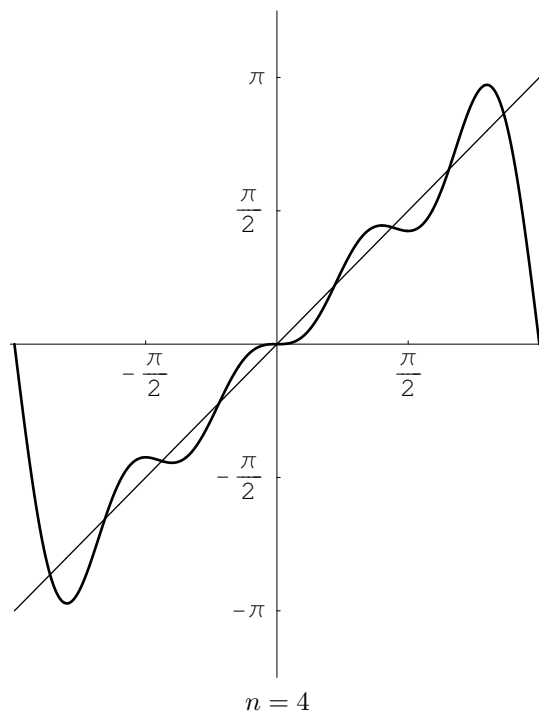
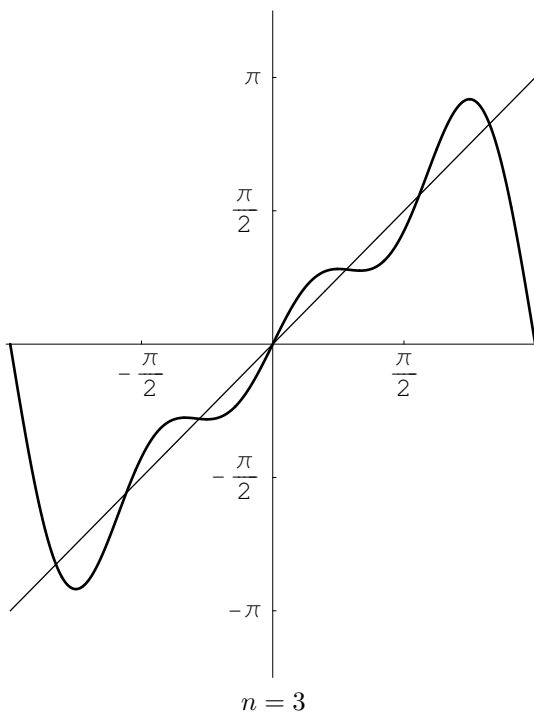
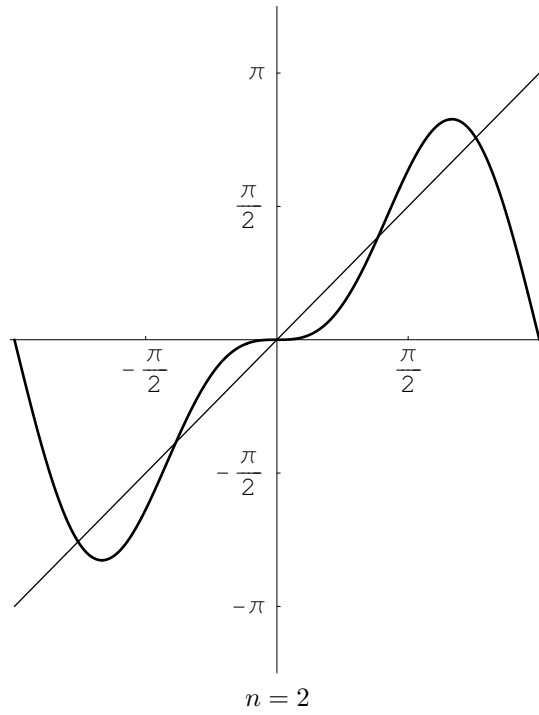
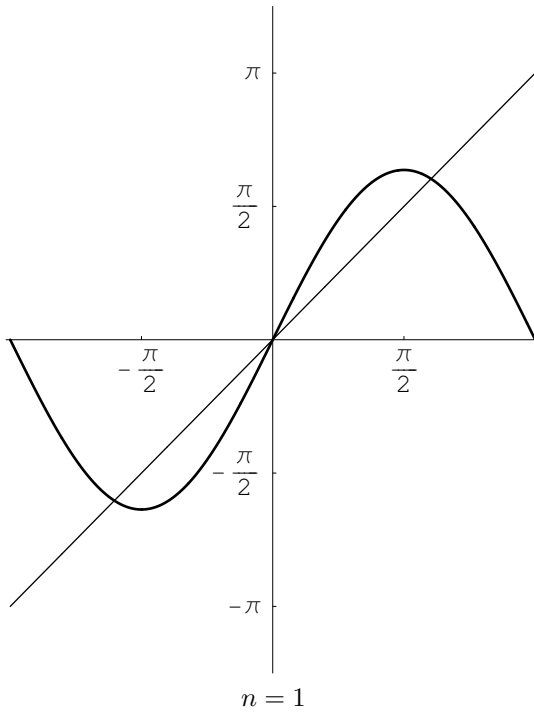
5. T from $\mathbb{R}^2$ to $\mathbb{C}$ defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + bi$ (over $\mathbb{R}$).

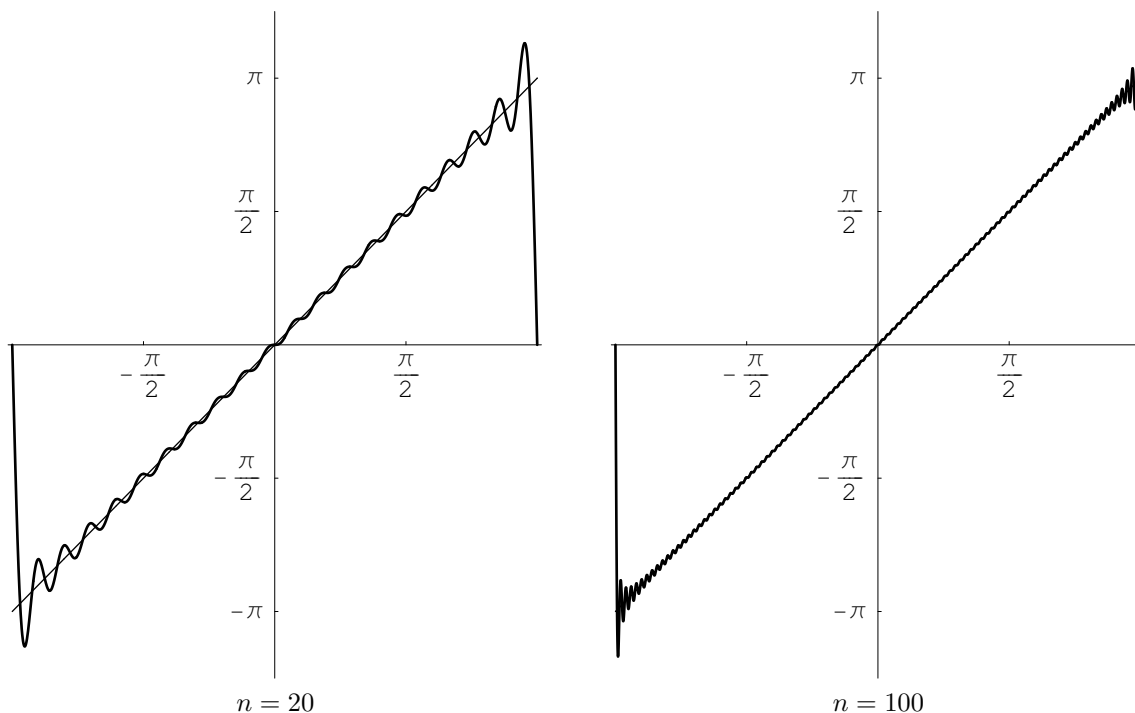
6. T from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$ defined by $T(f) = f' + \sin x$ (over $\mathbb{C}$).

7. I from $C^\infty(\mathbb{R})$ to $\mathbb{C}$ defined by $I(f) = \int_0^1 f(x) \, dx$ (over $\mathbb{C}$).

Fourier Series

Let $f(t) = t$. Here are the graphs of $f_n = \text{proj}_{T_n} f$ for various values of n .





Useful Trigonometric Identities

When finding Fourier series, you will sometimes need to integrate products of sines and cosines. There are some trigonometric identities which are useful for this; these identities essentially all come from the fact that $e^{i(A+B)} = e^{iA}e^{iB}$. Using the fact that $e^{i\theta} = \cos \theta + i \sin \theta$, we have

$$\begin{aligned}
 \cos(A+B) + i \sin(A+B) &= e^{i(A+B)} \\
 &= e^{iA}e^{iB} \\
 &= (\cos A + i \sin A)(\cos B + i \sin B) \\
 &= (\cos A \cos B - \sin A \sin B) + i(\sin A \cos B + \cos A \sin B)
 \end{aligned}$$

Equating the real and imaginary parts, we have

$$\begin{aligned}
 \cos(A+B) &= \cos A \cos B - \sin A \sin B \\
 \sin(A+B) &= \sin A \cos B + \cos A \sin B
 \end{aligned}$$

If we replace B by $-B$ and use the fact that $\cos(-B) = \cos B$ while $\sin(-B) = -\sin B$, we have

$$\begin{aligned}
 \cos(A-B) &= \cos A \cos B + \sin A \sin B \\
 \sin(A-B) &= \sin A \cos B - \cos A \sin B
 \end{aligned}$$

We can now use these to find identities for products. For instance, $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$, so

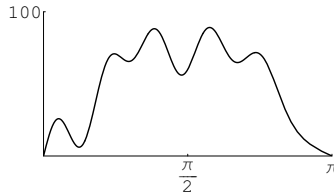
$$\sin A \cos B = \frac{1}{2} \sin(A+B) + \frac{1}{2} \sin(A-B),$$

which is a convenient formula if you need to integrate $\sin A \cos B$.

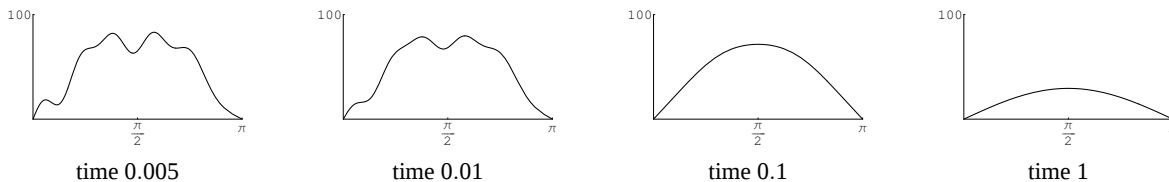
Partial Differential Equations

The Heat Equation

Imagine that we have a metal bar of length π which is thermally insulated (meaning that it doesn't lose heat to its surroundings). Suppose that, at time 0, the bar is heated so that a graph of the temperature along the bar looks like this:



Here, the x -axis represents length along the bar and the y -axis represents temperature. For example, the temperature at both ends of the bar is 0. Suppose that the ends of the bar are kept at temperature 0 while nothing is done to the rest of the bar. Since the ends of the bar are kept cold, we can guess that the temperature along the whole bar will eventually tend to 0. In fact, we will see that this is exactly what happens.



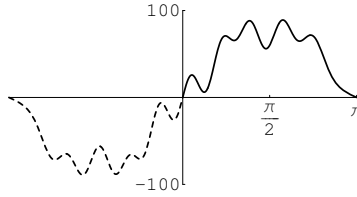
At any particular time t , we can represent the temperature along the bar as a function of x . So, we should really think of the temperature as a function of both time and position. Let $f(t, x)$ be the temperature of the bar at time t and position x . Since the ends of the bar are always kept at temperature 0, we know that $f(t, 0) = f(t, \pi) = 0$ for all t . Also, we are given a graph of $f(0, x)$ — the initial temperature distribution of the bar. In order to find out what $f(t, x)$ is at any time, we need to understand how the bar cools.

It turns out that the way the bar cools can be modeled by the heat equation $\frac{\partial f}{\partial t} = \mu \frac{\partial^2 f}{\partial x^2}$; here, μ is a positive constant indicating how well the bar conducts heat (metal would have a much larger μ than plastic). Since this differential equation involves partial derivatives, we call it a partial differential equation.

Fourier Series on $[0, \pi]$

We would like to use Fourier series to solve the heat equation. At any fixed time t , the temperature is just a function of x , so we should be able to write the Fourier series for that function. The only problem is that x goes from 0 to π rather than from $-\pi$ to π . That is, we have a function g on $[0, \pi]$ (representing the temperature at a given time), but we only know how to write Fourier series for functions on $[-\pi, \pi]$. To fix this problem, we simply extend g to a function $\tilde{g}$ on $[-\pi, \pi]$.

There are lots of ways to pick $\tilde{g}$; however, if we're smart about picking $\tilde{g}$, we can save ourselves a lot of work when we compute the Fourier series of $\tilde{g}$. Recall that the Fourier coefficients of $\tilde{g}$ are the inner products $\langle \tilde{g}(x), \frac{1}{\sqrt{2}} \rangle$, $\langle \tilde{g}(x), \sin(nx) \rangle$, and $\langle \tilde{g}(x), \cos(nx) \rangle$. Notice that $\frac{1}{\sqrt{2}}$ and $\cos(nx)$ are both even functions. In problem #2(c) of your Monday homework, you found that the inner product of an even function and an odd function is 0; thus, if we can make $\tilde{g}$ an odd function, then it will only have Fourier coefficients for the $\sin(nx)$ terms. Visually, here is how we extend g to an odd function $\tilde{g}$ on $[-\pi, \pi]$:



Now, $\langle \tilde{g}(x), \frac{1}{\sqrt{2}} \rangle = 0$ and $\langle \tilde{g}(x), \cos(nx) \rangle = 0$, so $\tilde{g}(x)$ has a Fourier series $\sum_{n=1}^{\infty} a_n \sin(nx)$ where $a_n = \langle \tilde{g}(x), \sin(nx) \rangle$. Since both $\tilde{g}$ and $\sin(nx)$ are odd functions, #2(b) of your Monday homework says that

$$a_n = \frac{2}{\pi} \int_0^{\pi} \tilde{g}(x) \sin(nx) dx.$$

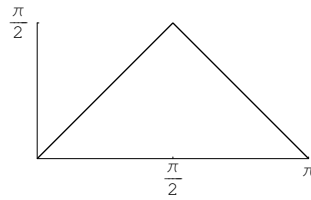
But on $[0, \pi]$, $\tilde{g}(x)$ is just $g(x)$! So, we really have the following.

Fact 1. If $g(x)$ is a (reasonable) continuous function on $[0, \pi]$ such that $g(0) = g(\pi) = 0$, then we can write $g(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ where

$$a_n = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx) dx.$$

Note: Without the condition $g(0) = g(\pi) = 0$, we can still write the Fourier series for g , but the Fourier series of g won't actually be equal to g . (The condition $g(0) = 0$ means that $\tilde{g}$ is continuous, and the condition $g(\pi) = 0$ means that $\tilde{g}(-\pi) = \tilde{g}(\pi)$, so $\tilde{g}$ is equal to its Fourier series.)

Example 2. Let $g(x)$ be the function on $[0, \pi]$ with the following graph.



That is,

$$g(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

In particular, $g(0) = g(\pi) = 0$. By Fact 1, we can write $g(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ where

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx) dx \\ &= \frac{2}{\pi} \left(\int_0^{\pi/2} x \sin(nx) dx + \int_{\pi/2}^{\pi} (\pi - x) \sin(nx) dx \right) \\ &= \frac{2}{\pi} \left(\left[-\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right]_0^{\pi/2} + \left[\frac{(x - \pi) \cos(nx)}{n} - \frac{\sin(nx)}{n^2} \right]_{\pi/2}^{\pi} \right) \\ &= \frac{2}{\pi} \left(\frac{2 \sin(\frac{n\pi}{2})}{n^2} - \frac{\sin(n\pi)}{n^2} \right) \\ &= \frac{4 \sin(\frac{n\pi}{2})}{\pi n^2} \text{ because } \sin(n\pi) = 0 \text{ for all } n \end{aligned}$$

Therefore,

$$g(x) = \sum_{n=1}^{\infty} \frac{4 \sin(\frac{n\pi}{2})}{\pi n^2} \sin(nx).$$

(Again, remember this statement is only true because $g(0) = g(\pi) = 0$.)

❖

Back to the Heat Equation

Now, we're ready to solve the heat equation. Remember that we were given a function representing the temperature at time 0; let's call this function $g(x)$. We were looking for a function $f(t, x)$ such that $\frac{\partial f}{\partial t} = \mu \frac{\partial^2 f}{\partial x^2}$, $f(t, 0) = f(t, \pi) = 0$, and $f(0, x) = g(x)$.

At any fixed time t , $f(t, x)$ is just a function of x on $[0, \pi]$ satisfying $f(t, 0) = f(t, \pi) = 0$. Therefore, by Fact 1, we can write

$$f(t, x) = \sum_{n=1}^{\infty} a_n(t) \sin(nx) \quad (1)$$

for some $a_n(t)$. Then,

$$\begin{aligned} \frac{\partial f}{\partial t}(t, x) &= \sum_{n=1}^{\infty} a'_n(t) \sin(nx) \\ \frac{\partial f}{\partial x}(t, x) &= \sum_{n=1}^{\infty} n a_n(t) \cos(nx) \\ \frac{\partial^2 f}{\partial x^2}(t, x) &= \sum_{n=1}^{\infty} (-n^2) a_n(t) \sin(nx) \end{aligned}$$

So, the equation $\frac{\partial f}{\partial t} = \mu \frac{\partial^2 f}{\partial x^2}$ can be rewritten as

$$\sum_{n=1}^{\infty} a'_n(t) \sin(nx) = \sum_{n=1}^{\infty} (-\mu n^2) a_n(t) \sin(nx).$$

Equating the coefficients of $\sin(nx)$ on each side, we have $a'_n(t) = -\mu n^2 a_n(t)$. This is a first-order linear differential equations whose solutions are $a_n(t) = c_n e^{-\mu n^2 t}$. Plugging this into (1), we obtain $f(t, x) = \sum_{n=1}^{\infty} c_n e^{-\mu n^2 t} \sin(nx)$. Plugging $t = 0$ into this equation gives $f(0, x) = \sum_{n=1}^{\infty} c_n \sin(nx)$. Thus, the constants c_n are simply the Fourier coefficients of the initial condition $f(0, x) = g(x)$. To summarize:

Fact 3. Consider the heat equation with initial conditions

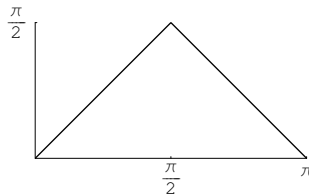
$$\begin{cases} \frac{\partial f}{\partial t} = \mu \frac{\partial^2 f}{\partial x^2} \\ f(t, 0) = f(t, \pi) = 0 \text{ for all } t \\ f(0, x) = g(x) \end{cases}$$

By Fact 1, we can write $g(x) = \sum_{n=1}^{\infty} c_n \sin(nx)$. Then, the solution of the heat equation is

$$f(t, x) = \sum_{n=1}^{\infty} c_n e^{-\mu n^2 t} \sin(nx).$$

Note: We've glossed over a lot of details — it actually takes quite a lot of math to make the above argument work!

Example 4. Let $f(t, x)$ be the temperature of a bar at time t and point x . Suppose the bar is initially heated so that its temperature distribution looks like this:



That is, the initial temperature is given by

$$g(x) = \begin{cases} x & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

The ends of the bar are kept at temperature 0 at all times, so $f(t, 0) = f(t, \pi) = 0$ for all t .

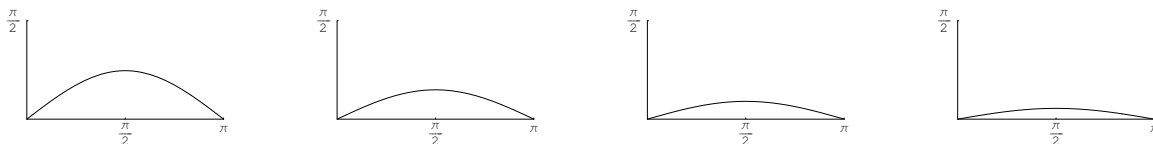
To find a formula for $f(t, x)$, we need to first write the Fourier series for $g(x)$. We did this in Example 2 and found that

$$g(x) = \sum_{n=1}^{\infty} \frac{4 \sin\left(\frac{n\pi}{2}\right)}{\pi n^2} \sin(nx).$$

Therefore, the solution of the heat equation is

$$f(t, x) = \sum_{n=1}^{\infty} \frac{4 \sin\left(\frac{n\pi}{2}\right)}{\pi n^2} e^{-\mu n^2 t} \sin(nx).$$

Here are graphs of $f(t, x)$ for $t = 1, 2, 3, 4$ (using $\mu = 0.5$).



In particular, it is clear from our formula that $f(t, x) \rightarrow 0$ as $t \rightarrow \infty$.

❖

Connection with Continuous Dynamical Systems

Earlier in the course, we looked at continuous dynamical systems of the form $\frac{d\vec{x}}{dt} = A\vec{x}$. As we saw, the solutions of these dynamical systems could be written in terms of the eigenvectors of A : if $\vec{v}_1, \dots, \vec{v}_m$ was an eigenbasis for A and $\lambda_1, \dots, \lambda_m$ were the corresponding eigenvalues, we found that the solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$ were $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + \dots + c_m e^{\lambda_m t} \vec{v}_m$. In particular, $\vec{x}(0) = c_1 \vec{v}_1 + \dots + c_m \vec{v}_m$. That is, the constants c_i came from writing the initial condition $\vec{x}(0)$ as a linear combination of the eigenvectors.

When we solve the heat equation $\frac{\partial f}{\partial t} = \mu \frac{\partial^2 f}{\partial x^2}$ using Fourier series, we are really doing something very similar. If we let D be the linear transformation $D(f) = \frac{\partial f}{\partial x}$, then the heat equation can be written as $\frac{\partial f}{\partial t} = \mu D^2 f$. Thus, μD^2 now plays the role of the matrix A , and we expect the eigenfunctions of μD^2 to figure into the solution. In fact, this is exactly what happens — the functions $\sin(nx)$ are eigenfunctions of μD^2 because $(\mu D^2)[\sin(nx)] = \mu D[n \cos(nx)] = -\mu n^2 \sin(nx)$. That is, $\sin(nx)$ is an eigenfunction of μD^2 with eigenvalue $-\mu n^2$. Now, compare $\frac{d\vec{x}}{dt} = A\vec{x}$ to $\frac{\partial f}{\partial t} = \mu \frac{\partial^2 f}{\partial x^2}$:

differential equation	$\frac{d\vec{x}}{dt} = A\vec{x}$	$\frac{\partial f}{\partial t} = \mu D^2 f$
linear transformation (LT)	A	μD^2
eigenvectors / eigenfunctions of the LT	$\vec{v}_i$	$\sin(nx)$
corresponding eigenvalues	λ_i	$-\mu n^2$
solutions of the differential equation	$\vec{x}(t) = \sum_{i=1}^m c_i e^{\lambda_i t} \vec{v}_i$	$f(t, x) = \sum_{n=1}^{\infty} c_n e^{-\mu n^2 t} \sin(nx)$
initial condition	$\vec{x}(0) = \sum_{i=1}^m c_i \vec{v}_i$	$f(0, x) = \sum_{n=1}^{\infty} c_n \sin(nx)$

As you can see, using Fourier series to solve partial differential equations is really exactly analogous to using eigenvectors to solve continuous dynamical systems!

The Wave Equation

Now, we'll look at a different partial differential equation. Suppose we have a string of length π whose ends are fixed but whose middle can move around (imagine a guitar string, for example). Let $f(t, x)$ be the height of point x of the string at time t . Then, f satisfies a differential equation $\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$, called the **wave equation**. Just as before, we can solve this partial differential equation by writing f as a Fourier series $f(t, x) = \sum_{n=1}^{\infty} r_n(t) \sin(nx)$. Differentiating,

$$\begin{aligned}\frac{\partial f}{\partial t}(t, x) &= \sum_{n=1}^{\infty} r'_n(t) \sin(nx) \\ \frac{\partial^2 f}{\partial t^2}(t, x) &= \sum_{n=1}^{\infty} r''_n(t) \sin(nx) \\ \frac{\partial f}{\partial x}(t, x) &= \sum_{n=1}^{\infty} n r_n(t) \cos(nx) \\ \frac{\partial^2 f}{\partial x^2}(t, x) &= \sum_{n=1}^{\infty} (-n^2) r_n(t) \sin(nx)\end{aligned}$$

Equating the Fourier coefficients of $\frac{\partial^2 f}{\partial t^2}$ and $c^2 \frac{\partial^2 f}{\partial x^2}$, we have $r''_n(t) = -n^2 c^2 r_n(t)$. This is a second-order linear differential equation, and its solutions are $r_n(t) = a_n \cos(nct) + b_n \sin(nct)$. Thus,

$$f(t, x) = \sum_{n=1}^{\infty} [a_n \cos(nct) + b_n \sin(nct)] \sin(nx) \quad (2)$$

for some constants a_n, b_n . Since the wave equation is a second-order equation in t , we need two initial conditions to get a unique solution. Usually, the initial conditions given are the initial position of the string, $f(0, x) = g(x)$, and the initial velocity of the string, $\frac{\partial f}{\partial t}(0, x) = h(x)$, which must satisfy $h(0) = h(\pi) = 0$. Plugging $t = 0$ into (2) gives

$$g(x) = f(0, x) = \sum_{n=1}^{\infty} a_n \sin(nx).$$

On the other hand, if we differentiate (2) and plug in $t = 0$, we get

$$h(x) = \frac{\partial f}{\partial t}(0, x) = \sum_{n=1}^{\infty} n c b_n \sin(nx).$$

Therefore, the a_n are the Fourier coefficients of $g(x)$ and the $n c b_n$ are the Fourier coefficients of $h(x)$. To summarize:

Fact 5. Consider the wave equation with initial conditions

$$\begin{cases} \frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2} \\ f(t, 0) = f(t, \pi) = 0 \text{ for all } t \\ f(0, x) = g(x) \\ \frac{\partial f}{\partial t}(0, x) = h(x) \text{ where } h(0) = h(\pi) = 0 \end{cases}$$

By Fact 1, we can write

$$g(x) = \sum_{n=1}^{\infty} a_n \sin(nx), h(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

Then, the solution of the wave equation is

$$f(t, x) = \sum_{n=1}^{\infty} \left[a_n \cos(nct) + \frac{b_n}{nc} \sin(nct) \right] \sin(nx).$$

In particular, we see that the constant c determines the period of oscillation of the string.

Example 6. Let $f(t, x)$ be the height of a guitar string at time t and point x . Suppose the string initially has height $g(x) = \sin x + \sin 3x - 2 \sin 4x$ and velocity $h(x) = -\sin x + \sin 3x$.

Notice that $g(x)$ and $h(x)$ are already written as Fourier series:

- $g(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ where $a_1 = 1$, $a_2 = 0$, $a_3 = 1$, $a_4 = -2$, and $a_n = 0$ for $n \geq 5$.
- $h(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ where $b_1 = -1$, $b_2 = 0$, $b_3 = 1$, and $b_n = 0$ for $n \geq 4$.

Therefore, the height of the string at any time is given by

$$f(t, x) = \left[\cos(ct) - \frac{1}{c} \sin(ct) \right] \sin(x) + \left[\cos(3ct) + \frac{1}{3c} \sin(3ct) \right] \sin(3x) + [-2 \cos(4ct)] \sin(4x).$$

Here are graphs of $f(t, x)$ for $t = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \dots, \frac{9\pi}{5}$ (using $c = 1$).

