11/01/2005 MIDTERM EXAMINATION   Math 21b

Name:

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- Start by printing your name in the above box and check the section in which you are.

- Try to answer each question on the same page as the question is asked. If needed, use the back or next empty page for work. If you need additional paper, write your name on it.

- Do not detach pages from this exam packet or unstaple the packet.

- Please write neatly. Answers which are illegible for the grader can not be given credit. Justify your answers.

- No notes, books, calculators, computers or other electronic aids are allowed.

- You have 120 minutes (2 hours) time to complete your work.

1		14
2		20
3		20
4		26
5		20
Total:		100

Problem (1) TF questions (14 points) Circle the correct letter and provide a brief justification. (Each problem gives 1 point for a correct answer and 1 point for a correct justification.)

(a) There is a linear transformation T satisfying $T\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and

$$T\begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -9 \end{bmatrix}.$$

T	F

Answer: False. Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $w_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$. We are given $T(v_1) = w_1$. We also know $5v_1 = \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix}$. $T(5v_1) = 5T(v_1) = 5(w_1)$ but $\begin{pmatrix} 5 \\ 0 \\ -9 \end{pmatrix} \neq \begin{pmatrix} 5 \\ 0 \\ -10 \end{pmatrix}$. So, T cannot be linear.

(b) If two nonzero vectors are linearly dependent, then each of them is a scalar multiple of the other.

T	F

Answer: True. By definition the vectors are collinear and thus scalar multiples.

(c) If two vectors are linearly dependent, then each of them is a scalar multiple of the other. (Hint: Notice one word is missing compared to the statement in (b).)

T	F

Answer: False. $\vec{0}$ is linearly dependent with all other vectors, but no scalar multiple of $\vec{0}$ will yield a non-zero vector.

| T | F | (d) If a 3×3 matrix A represents the projection onto a plane in $\mathbb{R}^3$, then the rank of A is 2. |

Answer: True. im(A) is plane and rank(A)=dim(im(A))=2.

| T | F | (e) There exists a 2×2 matrix B such that $B^4 = I_2$, where I_2 is the identity matrix. |

Answer: True. $B = I_2$ works.

| T | F | (f) The composition of a horizontal shear with a vertical shear is the identity matrix. |

Answer: False. Not true in general (pick any two non-trivial shears to check).

| T | F | (g) The nullity of a matrix A plus the rank of A equals the number of columns of A. |

Answer: True. This is precisely the Rank-Nullity Theorem.

Problem (2) (20 points)

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 0 & -5 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(a) (5 points) Find rref(A).

(b) (5 points) Find a vector $\vec{b}$ such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions, or explain why there is no such vector.

(c) (5 points) Find a vector $\vec{b}$ such that the system $A\vec{x} = \vec{b}$ has exactly one solution, or explain why there is no such vector.

(d) (5 points) Find a vector $\vec{b}$ such that the system $A\vec{x} = \vec{b}$ has no solutions, or explain why there is no such vector.

Answers:

(a) Add row I to III, Add II to I, Divide II by -1, Subtract II from III, Swap IV and III to get rref(A)=$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(b) Since nullity of A=1, the kernel spans a line so $\vec{b} = \vec{0}$ has infinite solutions.

(c) Since $rank(A) < 4$, A^{-1} does not exist, so there is no vector which would yield exactly one solution.

(d) Since one row of rref(A) has all zeroes, this $\vec{b} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ is possible to find. We can use an augmented

matrix to see what configurations will yield a contradiction. First, take $\begin{pmatrix} 1 & 1 & 0 & 2 & a \\ 0 & -1 & 0 & -5 & b \\ -1 & 0 & 0 & 3 & c \\ 0 & 0 & 1 & 3 & d \end{pmatrix}$ and

perform Gauss-Jordan elimination (add row I to III, II to I, Multiply II by -1, Subtract II from III) to get
$\begin{pmatrix} 1 & 0 & 0 & -3 & a+b \\ 0 & 1 & 0 & 5 & -b \\ 0 & 0 & 0 & 0 & a+b+c \\ 0 & 0 & 1 & 3 & d \end{pmatrix}$. Clearly if $a+b+c \neq 0$ a contradiction occurs, and we get an inconsistent

system. One such $\vec{b}$ is $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

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Problem (3) (20 points)

Consider

$$T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x} = A\vec{x}$$

(a) (5 points) Find the rank of A.

(b) (5 points) Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that the $\mathcal{B}$-matrix of the linear transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

or explain why that is not possible.

(c) (5 points) Find a basis $\mathcal{B}'$ of $\mathbb{R}^2$ such that the $\mathcal{B}'$-matrix B' of the linear transformation T is upper triangular or explain why that is not possible.

(d) (5 points) Express $\vec{v} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ in the basis $\mathcal{B}$ given by vectors $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Answers:

(a)

$$\det \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = 0 + 1 = 1 \neq 0$$

It follows that the rank of A is 2. Another way to see that is to compute rref(A), which is the identity.

(b) No such basis exists. Here is why:

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

This means $\mathcal{B}$-matrix of T is not invertible, whereas A is invertible since it has full rank. Similar matrices must either be both invertible or both non-invertible. Another way to solve this problem was to look at SAS^{-1} for some general

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and show that the resulting system in a, b, c, and d had no solution.

(c) No such basis exists. The transformation T is a clockwise rotation by 90 degrees (check how it maps the standard vectors $\vec{e_1}$ and $\vec{e_2}$). If B' is upper triangular, then it equals

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

Assume there exists some basis $\mathcal{B} = \{\vec{v_1}, \vec{v_2}\}$ in which T has matrix B'.

$$B'\vec{v_1} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} = a\vec{v_1}$$

For a rotation, there does not exist a vector $\vec{v_1}$ with such property. As for part (b), another way to solve this problem was to look at SAS^{-1} for some general

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and show that the resulting system in a, b, c, and d had no solution.

(d) Set up the system $c_1\vec{v_1} + c_2\vec{v_2} = \vec{v}$, which leads to

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

There is an unique solution given by $c_1 = -10$ and $c_2 = 4$.

Problem (4) (26 points)

(a) (5 points) Find A, the matrix of projection in $\mathbb{R}^2$ onto the line $y = 3x$.

(b) (5 points) Find B, the matrix of a counterclockwise rotation by 90 degrees also in $\mathbb{R}^2$.

(c) (5 points) Find the rank of ABA.

(d) (5 points) Find the rank of of BAA.

(e) (6 points) Let

$$D = \begin{bmatrix} 3 & 4 & 2 & 1 & 3 & 5 \\ 2 & 5 & 1 & 4 & 2 & 6 \\ 5 & 7 & 1 & 6 & 5 & 9 \\ 4 & 9 & 3 & 8 & 4 & 8 \\ 9 & 2 & 4 & 1 & 9 & 7 \\ 1 & 8 & 5 & 2 & 1 & 4 \end{bmatrix}$$

Is D invertible? Justify your answer. (Hint: there is no computation necessary).

Answers:

(a) Unit vector in direction of line is $\begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$ so if $u_1 = 1/\sqrt{10}$ and $u_2 = 3/\sqrt{10}$, $A = \begin{pmatrix} (u_1)^2 & (u_1 u_2) \\ (u_1 u_2) & (u_2)^2 \end{pmatrix} = \begin{pmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{pmatrix}$.

(b) $B = \begin{pmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(c) Rank(ABA)=dim(im(ABA)), and the image of ABA is just the result of projecting onto $y = 3x$, rotating by 90 degrees, and projecting back onto $y = 3x$. This will have dim=0 since the rotation will be perpendicular to the line. Could also multiply out the matrices directly.

(d) Since AA=A (A is a projection), rank(BAA)=rank(BA)=dim(im(BA)), but this is just $y = 3x$ rotated by 90 degrees. Therefore the image of BAA is a line, and dim(im(BAA))=1=rank(BAA).

(e) D is not invertible since the first and fifth columns are the same and therefore not linearly independent.

Let V be a subspace of $\mathbb{R}^3$ such that the matrix of $proj_V$ is given by

$$\begin{bmatrix} \frac{4}{49} & \frac{12}{49} & \frac{6}{49} \\ \frac{12}{49} & \frac{36}{49} & \frac{18}{49} \\ \frac{6}{49} & \frac{18}{49} & \frac{9}{49} \end{bmatrix}$$

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(a) (5 points) What is dim V?

(b) (5 points) Find a basis of V.

(c) (5 points) Find a basis of $V^\perp$.

(d) (5 points) Find the matrix of $proj_{V^\perp}$.

Answers:

(a)

$$V = Im\, proj_V = span \left(\begin{pmatrix} \frac{4}{49} \\ \frac{12}{49} \\ \frac{6}{49} \end{pmatrix}, \begin{pmatrix} \frac{12}{49} \\ \frac{36}{49} \\ \frac{18}{49} \end{pmatrix}, \begin{pmatrix} \frac{6}{49} \\ \frac{18}{49} \\ \frac{9}{49} \end{pmatrix} \right) = span \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}.$$

This means V is a line, so $dim\, V = 1$.

(b) $V = span \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$

(c) $dim V^\perp = 3 - 1 = 2$, so we need a basis of two vectors. Notice that $V^\perp$ is given by all vectors perpendicular to $\vec{w} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$. We are then looking for two such vectors, $\vec{v_1}$ and $\vec{v_2}$ such that $\vec{w} \cdot \vec{v_1} = 0$ and $\vec{w} \cdot \vec{v_2} = 0$.

Take for example

$$\vec{v_1} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

and

$$\vec{v_2} = \vec{w} \times \vec{v_1} = \begin{pmatrix} 3 \\ 9 \\ -20 \end{pmatrix}$$

.

(d) To project we need an orthonormal basis for the space we project upon. The basis found in (c) is orthogonal but not orthonormal. We fix that by normalizing it.

$$\vec{u_1} = \frac{\vec{v_1}}{||\vec{v_1}||} = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{u_2} = \frac{\vec{v_2}}{||\vec{v_2}||} = \frac{1}{7\sqrt{10}} \begin{pmatrix} 3 \\ 9 \\ -20 \end{pmatrix}$$

$$proj_{V^\perp} \vec{x} = \vec{x}^\parallel = (\vec{x} \cdot \vec{u_1}) \vec{u_1} + (\vec{x} \cdot \vec{u_2}) \vec{u_2}.$$

Plugging in $\vec{u_1}$ and $\vec{u_2}$ and gathering terms gives the matrix we are searching. Clearly, any orthonormal basis for $V^\perp$ does the trick. A common fallacy was to write down a matrix whose image was $V^\perp$, which is easier and incorrect.