Review 12.19.05 Math21b Christopher Phillips

1.If $\vec{x}(t+1) = A\vec{x}(t)$, where $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $A = \begin{pmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{pmatrix}$, find a closed formula for $\vec{x}(t+1)$. Is the zero state asymptotically stable?

2. An ecological system consists of two species whose populations at time t are given by $x(t)$ and $y(t)$. The evolution of the system is described by the equation $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(x-y+1) \\ y(y+x-3) \end{pmatrix}$. In the first quadrant, find all equilibrium points of this system and sketch the vector field (indicating nullclines). Are there any stable equilibrium points? If both species start with positive populations, can either become extinct?

3. Find all solutions to the differential equation $f''(t) - 2f'(t) + f(t) = 4e^{3t}$. Find the unique solutions given the initial conditions $f(0) = 1$ and $f'(0) = 1$.

4.The ends of a copper wire of length π are heated so that their temperatures are $T(t,0) = t$ and $T(t,\pi) = t + \frac{\pi^2}{2}$, respectively. The temperature of the wire at time $t = 0$ is given by $T(0,x) = sin(x) + \frac{x^2}{2}$. Assuming that $T(t,x)$ satisfies the heat equation $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$, the special case when $\mu = 1$, find $T(t,x)$ for all times $t \geqslant 0$ and all points on the wire $0 \geqslant x \geqslant \pi$. Hints: Find a particular solution $S(t,x)$, such that $T(t,0) - S(t,0) = 0$ and $T(t,\pi) - S(t,\pi) = 0$. There is such a solution $S(t,x)$ which is a polynomial in the variables t and x. Then use the linearity of solutions to find $T(t,x)$.

5. Find the Fourier series for $\cos(\frac{t}{2})$. You may find useful the fact that $\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$. Once you have done this, find a closed formula for $\sum\limits_{n=1}^{\infty} \frac{1}{4n^2-1}$.

Answers: 1. $\vec{x}(t) = \begin{pmatrix} -\sin(\theta t) \\ \cos(\theta t) \end{pmatrix}$, where θ is $\arctan \frac{4}{3}$ and the zero state is not asymptotically stable (the trajectory is an ellipse/circle).

2. The points (1,2), (0,0), and (0,3) are equilibrium points but none are stable. If both species start with positive populations, neither will go extinct.

3. The general solution are of the form $f(t) = c_1 e^t + c_2 t e^t + 3^{3t}$ and the particular solution is $f(t) = -2te^t + e^{3t}$.

4. $T(t, x) = e^{-t} \sin(x) + t + \frac{x^2}{2}$

5. $\cos(\frac{t}{2}) = \frac{2}{\pi} + \sum_{n=1}^{\infty} n = \frac{4(-1)^{n+1} \cos(nt)}{\pi(4n^2-1)}$. By letting $t = \pi$, we find that $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$.