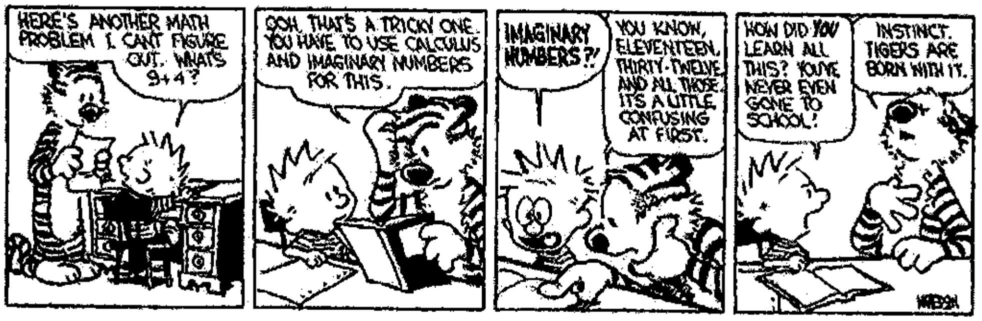
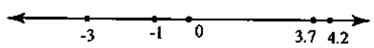
Complex Numbers



Basic Definitions

You're used to real numbers and how they're graphed on the real number line.

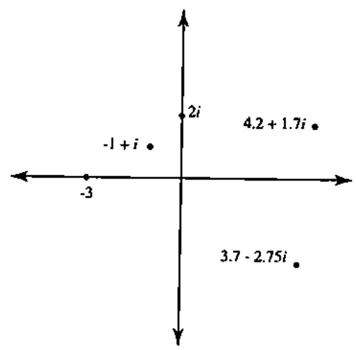


Real numbers are very useful, but there is one major problem we encounter when working with them: not all polynomials have real roots. For example, although the polynomials $x^2 - 3$ and $x^2 + 1$ look almost identical, the former has two real roots ($\pm\sqrt{3}$) while the latter has none. As we saw in class, this means that some matrices don't have any real eigenvalues. To fix this problem, mathematicians simply define a new number, called i, to be the square root of -1. Of course, i is not a real number, since there is no real number whose square is -1; instead, we call i an imaginary number. Now, the polynomial $x^2 + 1$ has two roots, i and $-i$.

More generally, an imaginary number is any real number multiple of i, like 0, $2i$, $-2.75i$, or $1.7i$. We can graph the imaginary numbers on a number line, but we use a vertical line instead of a horizontal line.

We know how to add and multiply real numbers, and we would like to do the same with imaginary numbers. However, if two imaginary numbers are multiplied, the answer is a real number; for instance, $(2i)(3i) = 6i^2 = -6$. This suggests that we shouldn't look at real and imaginary numbers separately; instead, we study complex numbers, which are sums of real and imaginary numbers. That is, a complex number is just a number of the form $x + iy$ where x and y are both real numbers. So, $1 + \sqrt{3}i$, $4.2 + 1.7i$, -3, $3.7 - 2.75i$, and $2i$ are all complex numbers. We write $\mathbb{C}$ for the set of complex numbers.

Since we view the set of real numbers as a horizontal line and the set of imaginary numbers as a vertical line, it's natural to view the set of complex numbers as a plane in which the real numbers lie on the horizontal axis and the imaginary numbers lie on the vertical axis. This plane is called the complex plane.



Arithmetic of Complex Numbers

As we said already, a complex number is just a number of the form $z = x + iy$ where $x, y \in \mathbb{R}$. We call x the real part of z and y the imaginary part of z. To add complex numbers, we just add the real and imaginary parts separately. That is, $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$.

Example 1. $(4.2 + 1.7i) + (3.7 - 2.75i) = (4.2 + 3.7) + (1.7 - 2.75)i = 7.9 - 1.05i$. ❖

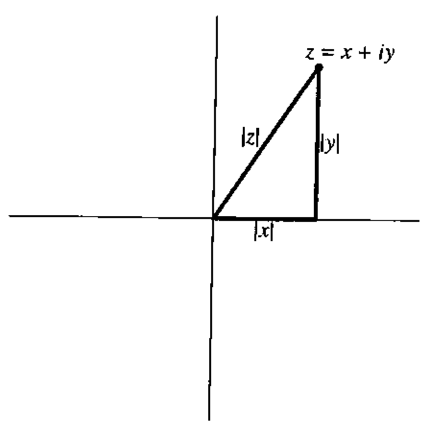
To multiply complex numbers, we use the distributive property. That is,

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 - y_1y_2 \text{ since } i^2 = -1 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)\end{aligned}$$

Example 2. $(1 + i)(5 + 3i) = 2 + 8i$. ❖

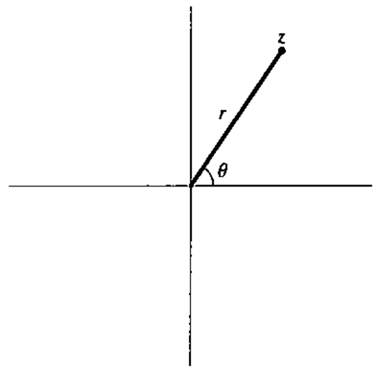
The Absolute Value of a Complex Number

When x is a real number, the absolute value of x measures the distance from x to 0 on the number line. For instance, 5 and -5 are both 5 units away from 0, so $|5| = |-5| = 5$. Similarly, the absolute value of a complex number $z = x + iy$ is defined to be the distance from z to 0. By the Pythagorean theorem, $|z| = \sqrt{x^2 + y^2}$.



Polar Coordinates

When we write a complex number z as $x + iy$ with $x, y \in \mathbb{R}$, we say that we are writing z in Cartesian coordinates. There is another useful way to write a complex number: any complex number z is determined by its distance from the origin (r) and an angle (θ):

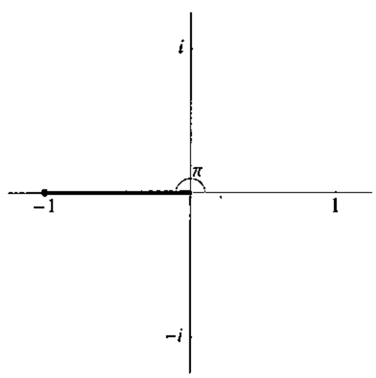


We call r and θ the polar coordinates of z. We can see from the diagram that $z = r\cos\theta + i(r\sin\theta)$. We write this more simply as $z = re^{i\theta}$. (If you want to know why $e^{i\theta} = \cos\theta + i\sin\theta$, try writing out the Taylor series of e^x, $\sin x$, and $\cos x$.)

To summarize, we can write a complex number z in the form $z = x + iy$ (Cartesian coordinates) or the form $z = re^{i\theta}$ (polar coordinates). These two forms are related as follows.

- $x = r\cos\theta$ and $y = r\sin\theta$.
- $r = |z| = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$.

Example 3. $e^{\pi i}$ is the complex number with $r = 1$ and $\theta = \pi$. By drawing a diagram, we see that $e^{\pi i} = -1$.

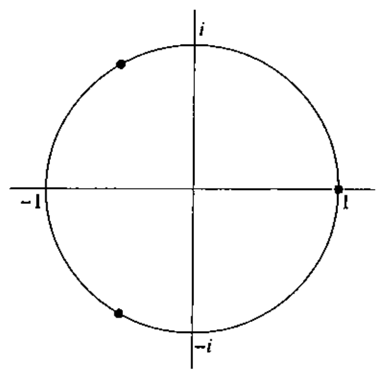


Similarly, $e^{2\pi i} = 1$. Notice that, if n is any integer, then $e^{2\pi i n} = 1$. ❖

If we are adding complex numbers, it is useful to write them in Cartesian coordinates; on the other hand, if we are multiplying complex numbers, it is often easier to use polar coordinates. After all, $\left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right)$ is just $r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$. Here are two examples of how polar coordinates can be useful.

Example 4. Let $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Suppose we want to find z^{100}. We could just start computing powers of z, but that would get boring really fast. Instead, let's write z in polar coordinates. We know that $z = re^{i\theta}$ where $r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$ and $\tan\theta = \sqrt{3}$. Thus, θ is either $\frac{\pi}{3}$ or $\frac{4\pi}{3}$. Since z lies in the first quadrant of the complex plane, it must be the case that $\theta = \frac{\pi}{3}$, so $z = e^{\pi i/3}$. Then, $z^{100} = \left(e^{\pi i/3}\right)^{100} = e^{100\pi i/3}$. In Cartesian coordinates, $z^{100} = \cos\frac{100\pi}{3} + i\sin\frac{100\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. ❖

Example 5. Let's find all complex numbers z such that $z^3 = 1$. We know we can write any complex number z as $re^{i\theta}$ for some r and θ. Then, $z^3 = r^3e^{3i\theta}$, so $z^3 = 1$ if and only if $r^3 = 1$ and 3θ is a multiple of 2π. So, the solutions of $z^3 = 1$ are $z = 1$, $e^{2i\pi/3}$, and $e^{4i\pi/3}$. These lie on a circle of radius 1 in the complex plane:



In Cartesian coordinates, $e^{2\pi i/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $e^{4\pi i/3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. ❖

The Fundamental Theorem of Algebra

As we discussed earlier, not all polynomials have real roots, so not all matrices have real eigenvalues. For complex numbers, the situation is much better.

The Fundamental Theorem of Algebra. *Let $p(x)$ be a polynomial whose coefficients are complex numbers (or just real numbers), and let n be the degree of $p(x)$. Then, $p(x)$ factors as $p(x) = c(x-a_1)\cdots(x-a_n)$ for some complex numbers $c, a_1, \ldots, a_n$. In particular, $p(x)$ has exactly n roots $a_1, \ldots, a_n$ (if the roots are counted with multiplicity).*

Example 6. The polynomial x^2+1 can be written as $(x-i)(x+i)$. That is, it has two roots, i and $-i$. ❖

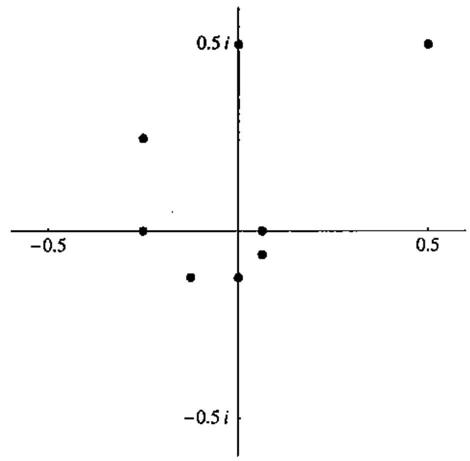
Example 7. The polynomial $x^2 + 2ix - 1$ can be written as $(x+i)^2$, so $-i$ is a root of multiplicity 2. ❖

The Fundamental Theorem of Algebra tells us that every $n \times n$ matrix has *exactly* n complex eigenvalues. Don't worry if you don't understand the theorem perfectly; we will talk about it more in class.

Practice Problems

1. *Let $z = \frac{1}{2} + \frac{1}{2}i$. Write z in polar coordinates. Draw $z, z^2, z^3, \ldots, z^8$ in the complex plane. Write z^5 in Cartesian coordinates.*

   Solution. We can write $z = re^{i\theta}$ where $r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$ and $\tan\theta = 1$. Since z is in the first quadrant of the complex plane, θ must be $\frac{\pi}{4}$. Thus, $z = \frac{1}{\sqrt{2}}e^{\pi i/4}$, so $z^n = \frac{1}{(\sqrt{2})^n}e^{n\pi i/4}$. Therefore, $z, z^2, \ldots, z^8$ form a sort of spiral inwards:



In particular,

$$\begin{aligned} z^5 &= \frac{1}{(\sqrt{2})^5} e^{5\pi i/4} \\ &= \frac{1}{4\sqrt{2}} e^{5\pi i/4} \\ &= \frac{1}{4\sqrt{2}} \cos\frac{5\pi}{4} + \left(\frac{1}{4\sqrt{2}} \sin\frac{5\pi}{4}\right) i \\ &= -\frac{1}{8} - \frac{1}{8} i \end{aligned}$$

2. *Explain why $\mathbb{C}$ is a linear space. What is its dimension?*

   Solution. Adding two complex numbers gives us another complex number; after all, we said that $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$; thus, $\mathbb{C}$ is closed under addition.

   If $z = x + iy$ is a complex number and c is a real number (i.e., a scalar), then $cz = cx + i(cy)$ is a complex number, so $\mathbb{C}$ is closed under scalar multiplication.

   Therefore, $\mathbb{C}$ is a linear space. Any element of $\mathbb{C}$ can be written uniquely as $x + iy = x \cdot 1 + y \cdot i$, so 1 and i form a basis of $\mathbb{C}$. Therefore, $\mathbb{C}$ has dimension 2 (which makes sense intuitively since $\mathbb{C}$ looks like a plane).

3. *If $z = re^{i\theta}$ is a nonzero complex number, write $\frac{1}{z}$ in polar coordinates.*

   Solution. If $z = re^{i\theta}$, then $\frac{1}{z} = \frac{1}{r}\frac{1}{e^{i\theta}} = \frac{1}{r}e^{-i\theta}$.

4. *Find the square roots of i.*

   Solution. We want to find $z = re^{i\theta}$ such that $z^2 = i$. In polar coordinates, $i = e^{\pi i/2}$, so we want $r^2 e^{2i\theta} = e^{\pi i/2}$. Therefore, we must have $r = 1$ and 2θ equal to $\frac{\pi}{2}$ plus a multiple of 2π. Thus, the square roots of i are $e^{\pi i/4}$ and $e^{5\pi i/4}$. In Cartesian coordinates, the square roots are $\pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$.