Coordinates

1. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$. Then, $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ is a basis of $\mathbb{R}^2$. Let L be the line $y = 2x$.

   (a) *Find* $[\vec{e}_1]_\mathfrak{B}$.

   Solution. We want to write $\vec{e}_1$ as $c_1\vec{v}_1 + c_2\vec{v}_2$. That is, we want to solve the system

$$\begin{bmatrix} 1 & 6 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   for c_1, c_2. Using Gauss-Jordan elimination, we have

$$\left[\begin{array}{cc:c} 1 & 6 & 1 \\ 2 & -3 & 0 \end{array}\right] \begin{array}{c} \\ -2(\mathrm{I}) \end{array} \rightarrow \left[\begin{array}{cc:c} 1 & 6 & 1 \\ 0 & -15 & -2 \end{array}\right] \begin{array}{c} \\ \div(-15) \end{array}$$

$$\rightarrow \left[\begin{array}{cc:c} 1 & 6 & 1 \\ 0 & 1 & 2/15 \end{array}\right] -6(\mathrm{II})$$

$$\rightarrow \left[\begin{array}{cc:c} 1 & 0 & 1/5 \\ 0 & 1 & 2/15 \end{array}\right]$$

   Thus, $c_1 = \frac{1}{5}$ and $c_2 = \frac{2}{15}$, so $\vec{e}_1 = \frac{1}{5}\vec{v}_1 + \frac{2}{15}\vec{v}_2$, and $[\vec{e}_1]_\mathfrak{B} = \begin{bmatrix} 1/5 \\ 2/15 \end{bmatrix}$.

   (b) *Find a matrix S such that $\vec{x} = S[\vec{x}]_\mathfrak{B}$ for all $\vec{x} \in \mathbb{R}^2$.*

   Solution. Observe that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, [\vec{v}_1]_\mathfrak{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 6 \\ -3 \end{bmatrix}, [\vec{v}_2]_\mathfrak{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

   Therefore, if we let S be the matrix

$$S = \begin{bmatrix} 1 & 6 \\ 2 & -3 \end{bmatrix},$$

   then $\vec{v}_1 = S[\vec{v}_1]_\mathfrak{B}$ and $\vec{v}_2 = S[\vec{v}_2]_\mathfrak{B}$. Since $\vec{v}_1$ and $\vec{v}_2$ form a basis of $\mathbb{R}^2$, it follows that $\vec{x} = S[\vec{x}]_\mathfrak{B}$ for any $\vec{x} \in \mathbb{R}^2$.

   (c) Find the $\mathfrak{B}$-matrix of proj_L.

   Solution. Since $\vec{v}_1$ lies on the line L, $\mathrm{proj}_L(\vec{v}_1) = \vec{v}_1 = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2$. Since $\vec{v}_2$ is perpendicular to L, $\mathrm{proj}_L(\vec{v}_2) = \vec{0} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2$. In $\mathfrak{B}$-coordinates, we write

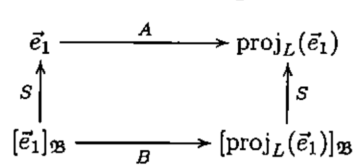
$$[\mathrm{proj}_L(\vec{v}_1)]_\mathfrak{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } [\mathrm{proj}_L(\vec{v}_2)]_\mathfrak{B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

   Therefore, the $\mathfrak{B}$-matrix of proj_L is

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

   (d) Use the $\mathfrak{B}$-matrix of proj_L to find the standard matrix A of proj_L.

   Solution. To write the standard matrix A of proj_L, we want to find $\mathrm{proj}_L(\vec{e}_1)$ and $\mathrm{proj}_L(\vec{e}_2)$. We can organize information into a commutative diagram:



(Here, B is the matrix we found in part (b).) The meaning of the vertical arrows is this: we showed in part (b) that $\vec{x} = S[\vec{x}]_{\mathfrak{B}}$ for any $\vec{x} \in \mathbb{R}^2$. In particular, $\vec{e}_1 = S[\vec{e}_1]_{\mathfrak{B}}$ and $\text{proj}_L(\vec{e}_1) = S[\text{proj}_L(\vec{e}_1]_{\mathfrak{B}}$. The top horizontal arrow indicates that $\text{proj}_L(\vec{e}_1) = A\vec{e}_1$ (since A is the standard matrix of proj_L). The bottom horizontal arrow means that $[\text{proj}_L(\vec{e}_1)]_{\mathfrak{B}} = B[\vec{e}_1]_{\mathfrak{B}}$; this is true since B is the $\mathfrak{B}$-matrix of proj_L.

By part (a), we know that

$$[\vec{e}_1]_{\mathfrak{B}} = \begin{bmatrix} 1/5 \\ 2/15 \end{bmatrix}.$$

In other words, $\vec{e}_1 = \frac{1}{5}\vec{v}_1 + \frac{2}{15}\vec{v}_2$. Using our diagram, we see that

$$[\text{proj}_L(\vec{e}_1)]_{\mathfrak{B}} = B[\vec{e}_1]_{\mathfrak{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/5 \\ 2/15 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 0 \end{bmatrix}.$$

That is, $\text{proj}_L(\vec{e}_1) = \frac{1}{5}\vec{v}_1$. Finally, we want to write $\text{proj}_L(\vec{e}_1)$ in standard coordinates. We have

$$\text{proj}_L(\vec{e}_1) = S[\text{proj}_L(\vec{e}_1)_{\mathfrak{B}}] = \begin{bmatrix} 1 & 6 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}.$$

Thus, we have found the first column of the standard matrix of proj_L.

Using the exact same argument for $\vec{e}_2$, we find that $[\vec{e}_2]_{\mathfrak{B}} = \begin{bmatrix} 2/5 \\ -1/15 \end{bmatrix}$, $[\text{proj}_L(\vec{e}_2)]_{\mathfrak{B}} = \begin{bmatrix} 2/5 \\ 0 \end{bmatrix}$, and $\text{proj}_L(\vec{e}_2) = \begin{bmatrix} 2/5 \\ 4/5 \end{bmatrix}$. Therefore, the standard matrix of proj_L is

$$\begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}.$$

2. *Let* $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ *and* $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. *Then,* $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ *is a basis of* $\mathbb{R}^2$. *Let* $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$ *and* $T(\vec{x}) = A\vec{x}$. *Find the matrix of* T *with respect to the basis* $\mathfrak{B}$.

Solution. There are different ways we could approach this.

- *Method 1: Construct the* $\mathfrak{B}$*-matrix column by column.*

  Remember that the columns of the $\mathfrak{B}$-matrix of T are $[T(\vec{v}_1)]_{\mathfrak{B}}$ and $[T(\vec{v}_2)]_{\mathfrak{B}}$. We have

$$T(\vec{v}_1) = A\vec{v}_1 = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$$

$$T(\vec{v}_2) = A\vec{v}_2 = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

  We can see by inspection that $T(\vec{v}_1) = 5\vec{v}_1 - 3\vec{v}_2$ and $T(\vec{v}_2) = 2\vec{v}_1 - 2\vec{v}_2$. So,

$$[T(\vec{v}_1)]_{\mathfrak{B}} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} \text{ and } [T(\vec{v}_2)]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

  Therefore, the $\mathfrak{B}$-matrix of T is

$$\begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}.$$

- *Method 2: Use the formula $B = S^{-1}AS$.*

  We defined a matrix S whose columns were the vectors in $\mathfrak{B}$. In this case,

  $$S = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

  Then, we showed that the $\mathfrak{B}$-matrix of $T(\vec{x}) = A\vec{x}$ is just $S^{-1}AS$. In this case (after some computation), we find that

  $$S^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix},$$

  so

  $$S^{-1}AS = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}.$$

3. Let P be the plane $x_1 + 2x_2 - 3x_3 = 0$. Any vector $\vec{x} \in \mathbb{R}^3$ can be written uniquely as a sum $\vec{x}^P + \vec{x}^\perp$ where $\vec{x}^P$ is a vector in the plane P and $\vec{x}^\perp$ is a vector perpendicular to P. We define a linear transformation proj_P from $\mathbb{R}^3$ to $\mathbb{R}^3$ by $\text{proj}_P(\vec{x}) = \vec{x}^P$. (This is very much like the projection transformations we talked about before, except that we are now projecting onto a plane rather than a line.)

   (a) *Find a convenient basis $\mathfrak{B}$ of $\mathbb{R}^3$ and write the $\mathfrak{B}$-matrix of* proj_P.

   Solution. If $\vec{v}$ lies in the plane P, then $\text{proj}_P(\vec{v})$ is simply $\vec{v}$. On the other hand, if $\vec{v}$ is perpendicular to the plane P, then $\text{proj}_P(\vec{v}) = \vec{0}$. Thus, we would like to find a basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ of $\mathbb{R}^3$ consisting of vectors which either lie in P or are perpendicular to P.

   We start by trying to find vectors lying in P. Notice that the equation $x_1 + 2x_3 - 3x_3 = 0$ can be written in matrix form as

   $$\begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad (*)$$

   In other words, the plane P is exactly the kernel of the matrix $\begin{bmatrix} 1 & 2 & -3 \end{bmatrix}$. To find the kernel of this matrix, we write the augmented matrix $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \end{array}\right]$. This is already in reduced row-echelon form, so we can just read the solutions off. The free variables are x_2 and x_3, so the solutions look like

   $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s + 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

   Thus, a basis of P is

   $$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

   On the other hand, by (*), the vector

   $$\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

   is perpendicular to P. Thus, $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a basis of $\mathbb{R}^3$. Moreover, since $\vec{v}_1$ and $\vec{v}_2$ lie in P, $\text{proj}_P(\vec{v}_1) = \vec{v}_1$ and $\text{proj}_P(\vec{v}_2)$. Since $\vec{v}_3$ is perpendicular to P, $\text{proj}_P(\vec{v}_3) = \vec{0}$. So, the $\mathfrak{B}$-matrix of proj_P is

   $$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) *Use your answer from part (a) to find the standard matrix of* proj_P.

Solution. If

$$S = \begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix},$$

the standard matrix of proj_P is SBS^{-1}. Since

$$S^{-1} = \begin{bmatrix} -1/7 & 5/7 & 3/7 \\ 3/14 & 3/7 & 5/14 \\ 1/14 & 1/7 & -3/14 \end{bmatrix},$$

we have

$$SBS^{-1} = \begin{bmatrix} 13/14 & -1/7 & 3/14 \\ -1/7 & 5/7 & 3/7 \\ 3/14 & 3/7 & 5/14 \end{bmatrix}.$$