Diagonalization

Let L be the line $y = 2x$ in $\mathbb{R}^2$. Let ref_L be reflection over L, and let A be the standard matrix of ref_L.

1. *Find an eigenbasis $\mathfrak{B}$ for A.*

   Solution. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Then, $\vec{v}_1$ lies on the line L, so $A\vec{v}_1 = \mathrm{ref}_L(\vec{v}_1) = \vec{v}_1$. On the other hand, $\vec{v}_2$ is perpendicular to L, so $A\vec{v}_2 = \mathrm{ref}_L(\vec{v}_2) = -\vec{v}_2$. Thus, $\vec{v}_1$ and $\vec{v}_2$ are both eigenvectors of A. Since $(\vec{v}_1, \vec{v}_2)$ is clearly a basis of $\mathbb{R}^2$, $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ is an eigenbasis for A.

2. *Find the $\mathfrak{B}$-matrix of* ref_L.

   Solution. If D is the $\mathfrak{B}$-matrix of ref_L, the columns of D are $[\mathrm{ref}_L(\vec{v}_1)]_\mathfrak{B}$ and $[\mathrm{ref}_L(\vec{v}_2)]_\mathfrak{B}$. Since $\mathrm{ref}_L(\vec{v}_1) = \vec{v}_1 = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2$, $[\mathrm{ref}_L(\vec{v}_1)]_\mathfrak{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since $\mathrm{ref}_L(\vec{v}_2) = -\vec{v}_2 = 0 \cdot \vec{v}_1 + (-1) \cdot \vec{v}_2$, $[\mathrm{ref}_L(\vec{v}_2)]_\mathfrak{B} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. So, the $\mathfrak{B}$-matrix of ref_L is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

3. *Find A (the standard matrix of* ref_L*).*

   Solution. If $S = [\vec{v}_1\ \vec{v}_2]$, then $A = SDS^{-1}$. In this case, this says that

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}.$$

True / False

1. *If A is diagonalizable, then A^2 is diagonalizable.*

   Solution. True. Since A is diagonalizable, there is an invertible matrix S such that $S^{-1}AS$ is a diagonal matrix. Then, $S^{-1}A^2S = (S^{-1}AS)(S^{-1}AS)$ is the product of two diagonal matrices, which is a diagonal matrix. Therefore, A^2 is diagonalizable.

2. *If A and B are $n \times n$ diagonalizable matrices, then $A + B$ is diagonalizable.*

   Solution. False. For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are both diagonalizable, but $A + B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not.

3. *If A and B are $n \times n$ diagonalizable matrices with the same eigenvectors, then AB is diagonalizable.*

   Solution. True. Since A is diagonalizable, there is an eigenbasis for A, say $(\vec{v}_1, \ldots, \vec{v}_n)$. Since B has the same eigenvectors as A, $(\vec{v}_1, \ldots, \vec{v}_n)$ is also an eigenbasis for B. Therefore, if $S = [\vec{v}_1\ \ldots, \vec{v}_n]$, $S^{-1}AS$ and $S^{-1}BS$ are both diagonal matrices. If we multiply two diagonal matrices, we get another diagonal matrix. Thus, $(S^{-1}AS)(S^{-1}BS) = S^{-1}ABS$ is a diagonal matrix, so AB is diagonalizable.

4. *If A is diagonalizable, then A^T is diagonalizable.*

   Solution. True. Since A is diagonalizable, there is an invertible matrix S such that $S^{-1}AS$ is a diagonal matrix D. Then, $(S^{-1}AS)^T$ is equal to D^T, which is the same as D. On the other hand, $(S^{-1}AS)^T$ is just $S^TA^T(S^{-1})^T = S^TA^T(S^T)^{-1}$. Thus, if $R = (S^T)^{-1}$, then $R^{-1}A^TR = D$, so A^T is diagonalizable.

5. *If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.*

   Solution. True. The geometric multiplicity of any eigenvalue is at least 1, so, if A has n distinct eigenvalues, then the sum of the geometric multiplicities of the eigenvalues is n. Therefore, A has a basis of eigenvectors, so A is diagonalizable.

6. *If A is a diagonalizable matrix and λ is an eigenvalue of A, then the algebraic multiplicity of λ is equal to the geometric multiplicity of λ.*

   Solution. True. Let's say A is an $n \times n$ matrix, and let λ be an eigenvalue. Then, we know:

   - The algebraic multiplicity of λ is greater than or equal to the geometric multiplicity of λ.
   - The sum of the algebraic multiplicities of all eigenvalues is at most n (the degree of the characteristic polynomial of A).
   - Since A is diagonalizable, the geometric multiplicities of the eigenvalues of A must add up to n.

   Thus, the algebraic multiplicity of λ must be the same as the geometric multiplicity of λ (otherwise the sum of the algebraic multiplicities would be greater than n).

7. *If A and B are both diagonalizable and if A and B have the same eigenvalues with the same geometric multiplicities, then A is similar to B.*

   Solution. True. Since A is diagonalizable, the geometric multiplicities of its eigenvalues add up to n. That is, A has n eigenvalues $\lambda_1, \ldots, \lambda_n$ (if we count the eigenvalues with their geometric multiplicities). Then, A is similar to the diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}.$$

   Since B has the same eigenvalues with the same geometric multiplicities and B is diagonalizable, B is also similar to D. Thus, A and B are both similar to D, so they must be similar to each other.