Eigenvectors

1. *True or false: if $(\vec{v}_1, \ldots, \vec{v}_n)$ is an eigenbasis for A, then $(\vec{v}_1, \ldots, \vec{v}_n)$ is an eigenbasis for A^2.*

   Solution. True. Since $(\vec{v}_1, \ldots, \vec{v}_n)$ is an eigenbasis for A, $(\vec{v}_1, \ldots, \vec{v}_n)$ is a basis of $\mathbb{R}^n$ and each $\vec{v}_i$ is an eigenvector of A. In particular, for each i, there is some scalar λ_i such that $A\vec{v}_i = \lambda_i\vec{v}_i$. Then, $A^2\vec{v}_i = A(A\vec{v}_i) = A(\lambda_i\vec{v}_i) = \lambda_i(A\vec{v}_i) = \lambda_i^2\vec{v}_i$, so $\vec{v}_i$ is an eigenvector of A^2. Thus, we have shown that $(\vec{v}_1, \ldots, \vec{v}_n)$ is a basis for $\mathbb{R}^n$ and that each $\vec{v}_i$ is an eigenvector of A^2. Therefore, $(\vec{v}_1, \ldots, \vec{v}_n)$ is an eigenbasis of A^2.

2. *Let V be a plane in $\mathbb{R}^3$ which contains the origin, and let A be the matrix of* proj_V *(that is,* $\text{proj}_V(\vec{x}) = A\vec{x}$*). Find the eigenvalues of A, and describe the eigenspaces geometrically. Is there an eigenbasis for A?*

   Solution. If $\vec{x} \in V$, then $A\vec{x} = \text{proj}_V(\vec{x}) = \vec{x}$, so $\vec{x}$ is an eigenvector of A with eigenvalue 1. E_1, the 1-eigenspace of A, consists of all vectors $\vec{x} \in \mathbb{R}^3$ such that $\text{proj}_V(\vec{x}) = \vec{x}$; this is simply the plane V.

   If $\vec{x} \in V^\perp$, then $A\vec{x} = \text{proj}_V(\vec{x}) = \vec{0}$, so $\vec{x}$ is an eigenvector of A with eigenvalue 0. E_0, the 0-eigenspace of A, consists of all vectors $\vec{x} \in \mathbb{R}^3$ such that $\text{proj}_V(\vec{x}) = \vec{0}$; this is simply the line $V^\perp$.

   There is an eigenbasis for A: if we choose two linearly independent vectors $\vec{v}_1, \vec{v}_2$ in E_0 and a nonzero vector $\vec{v}_3$ in E_1, then $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is an eigenbasis for A.

3. *Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, which represents a shear. Find the eigenvalues of A and their algebraic and geometric multiplicities. Is there an eigenbasis for A?*

   Solution. The characteristic polynomial of A is

$$f_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2.$$

   Thus, the only eigenvalue of A is 1, and it has algebraic multiplicity 2.

   The 1-eigenspace of A is $\ker(A - I)$, which is spanned by the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Therefore, 1 has geometric multiplicity 1, and there is no eigenbasis for A.

4. *Let A be a noninvertible $n \times n$ matrix. Explain why 0 must be an eigenvalue of A. Find the geometric multiplicity of the eigenvalue 0 in terms of* $\text{rank}(A)$.

   Solution. If A is not invertible, then $\det(A - 0I_n) = \det A = 0$, so 0 is an eigenvalue of A. The geometric multiplicity of 0 is $\dim[\ker(A - 0I_n)] = \dim(\ker A) = n - \text{rank}(A)$.

5. *Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then, A and B are similar. Find the characteristic polynomials, eigenvalues, and eigenvectors of A and B.*

   Solution. The characteristic polynomial of A is $\lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$, so the eigenvalues of A are 1 and -1. The 1-eigenspace of A is the kernel of $A - I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, which is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The -1-eigenspace of A is the kernel of $A + I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, which is spanned by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

   The characteristic polynomial of B is $(1-\lambda)(-1-\lambda) = \lambda^2 - 1$, which is the same as the characteristic polynomial of A. Therefore, B has the same eigenvalues, 1 and -1. The 1-eigenspace of B is the kernel of $B - I = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$, which is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The -1-eigenspace of B is the kernel of $B + I = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, which is spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thus, although A and B have the same characteristic polynomials and the same eigenvalues, they do not have the same eigenvectors.

6. *Is the matrix* $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *similar to the matrix* $\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$*?*

Solution. The two matrices have different traces, so they can't be similar.