The Gram-Schmidt Process

Let $M = \begin{bmatrix} 1 & 3 & 1 \\ 1 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ and $V = \operatorname{im} M$. Then, the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}$$

form a basis of V (do you see why?).

1. *Find an orthonormal basis of V.*

Solution. We start by letting $\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1$. Since $\vec{v}_1 \cdot \vec{v}_1 = 4$, $\|\vec{v}_1\| = 2$. So,

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

Since $\vec{u}_1 \cdot \vec{v}_2 = 2$,

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \\ 2 \end{bmatrix}.$$

Now, $\|\vec{v}_2^{\perp}\| = 4$, so we take

$$\vec{u}_2 = \frac{1}{4}\vec{v}_2^{\perp} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}.$$

Finally, $\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2$. Since $\vec{u}_1 \cdot \vec{v}_3 = 2$ and $\vec{u}_2 \cdot \vec{v}_3 = -2$,

$$\vec{v}_3^{\perp} = \vec{v}_3 - 2\vec{u}_1 + 2\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

Now, $\|\vec{v}_3^{\perp}\| = 2$, so

$$\vec{u}_3 = \frac{1}{2}\vec{v}_3^{\perp} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}.$$

Thus, an orthonormal basis of V is

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}.$$

Of course, it is easy to check that these vectors really are orthonormal.

2. *Find the QR-factorization of M.*

Solution. The matrix Q simply consists of the new orthonormal basis vectors; that is, $Q = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]$. So,

$$Q = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}.$$

Let $\mathfrak{A} = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$. Then, the matrix R has columns $[\vec{v}_1]_{\mathfrak{A}}$, $[\vec{v}_2]_{\mathfrak{A}}$, and $[\vec{v}_3]_{\mathfrak{A}}$.

Since $\mathfrak{A}$ is an orthonormal basis of V, recall that any $\vec{x} \in V$ can be written as $\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 + (\vec{u}_3 \cdot \vec{x})\vec{u}_3$. That is,

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} \vec{u}_1 \cdot \vec{x} \\ \vec{u}_2 \cdot \vec{x} \\ \vec{u}_3 \cdot \vec{x} \end{bmatrix}.$$

In particular,

$$[\vec{v}_1]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, [\vec{v}_2]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, [\vec{v}_3]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}.$$

Therefore,

$$R = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 4 & -2 \\ 0 & 0 & 2 \end{bmatrix}.$$

We can easily check that this is correct: the columns of Q are orthonormal, R is upper triangular with positive entries on the diagonal, and $M = QR$, which means that we really do have the correct QR-factorization.

3. *Find the QR-factorization of* $M = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$.

Solution. Rather than performing the Gram-Schmidt process on the columns of M, we can simply observe that

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

The first matrix has orthogonal columns while the second one is upper triangular with positive entries, so this must be the QR-factorization of M (because the QR-factorization is unique).